THE MESSENGER OF MATHEMATICS.

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MESSENGER OF MATHEMATICS.

4-TIC & 3-BIC RESIDUACITY-TABLES.

By *Lt.-Col. Allan Cunningham, R.E., and Thorold Gosset.*

1. *Introduction.* THE object of this Memoir is to introduce Tables for determining the 4-tic and 3-bic Residuacity of small prime Bases (q) and also that of the products of such Bases (q_1, q_2, ...) towards prime Moduli (p), *i.e.* to determine whether

$$q^{\frac{1}{4}(p-1)} \equiv, \text{ or not} \equiv -1;\ \text{or} \equiv, \text{ or not} \equiv +1 \pmod{p} \quad \text{.........(1)},$$

$$q^{\frac{1}{3}(p-1)} \equiv, \text{ or not} \equiv 1 \pmod{p} \quad \text{.........(2)}.$$

The Memoir is divided into three Chapters:—

CHAP. I. *Quartic Residuacity.* Art. 3—15*b*.
CHAP. II. *Cubic Residuacity.* Art. 16—29*b*.
CHAP. III. *Composite Bases and Examples.* Art. 30—33.
Tables A, B, and *Appendix.*

1a. *Pépin's Formulæ.* The expressions of the Laws of quartic and cubic reciprocity used in the present Memoir are those developed by Père Pépin in his Memoir quoted* below, referred to hereafter (for brevity) by the author's name only.

2. *Notation.* All symbols are *integers.*

q, Q the Bases whose 4-tic or 3-bic Residuacity is sought modulo p.

q any small prime (usually odd), Q any base.

p a prime$=4\varpi+1$, or $3\varpi+1$, the modulus of the operation.

π, π_1, π_2, &c., complex factors of p.

ρ a *proper root* of $\rho^4-1=0$, or $\rho^3-1=0$.

λ, μ are the solutions of the congruences which determine the sought residuacity of q modulo p.

ω means an *odd* number; ϵ means an *even* number.

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* *Mémoire sur les lois de réciprocité relatives aux résidus de puissances,* par le P. Th. Papin, S.J., Rome, 1878.

CHAP. I. *Quartic Residuacity.*

3. *Quartic Residuacity.* In order that Result (1)—the 4-tic residual relation—may be possible, it is clearly necessary that—

$$p \text{ must be of form } p = 4\varpi + 1 \ldots\ldots\ldots\ldots (3),$$

which involves—

$$p = a^2 + b^2, \text{ [a } odd, \text{ b } even \text{ (always reckoned +)]} \ldots\ldots (4);$$

$$\text{a is reckoned +, if a} = 4\alpha + 1; \text{ and } -, \text{ if a} = 4\alpha - 1] \ldots\ldots (4a),$$

which gives $p =$ product of *two* complex factors, say π, π', of type π, so that

$$p = \pi\pi', \text{ wherein } \pi = a + b\iota, \; \pi' = a - b\iota \ldots\ldots\ldots (5)$$

where ι is a *proper* root of $\rho^4 - 1 \equiv 0$, the roots of which are

$$\rho, \rho^2, \rho^3, \rho^4 = \iota, -1, -\iota, +1, \text{ so that } \iota^2 = -1, \text{ with } \iota = +\sqrt{-1} \ldots\ldots (6),$$

and the 4-tic residual relation of the Base q modulo π is

$$q^{\frac{1}{4}(p-1)} \equiv \text{ one of the roots } \rho \pmod{\pi}$$
$$\equiv \text{ one of } \iota, -1, -\iota, +1 \pmod{\pi} \ldots\ldots\ldots (7).$$

The symbol $(q/\pi)_4$ is commonly used as an *abbreviation*; thus

$$(q/\pi)_4 \text{ means Residue of } q^{\frac{1}{4}(p-1)} \text{ modulo } \pi \ldots\ldots\ldots (8).$$

It is immaterial which of the complex factors (π, π') is used; but it is essential that the *same* factor* (π, or π') should be used throughout: the effect of changing the factors (π, π') is to interchange the Residues $\pm\iota$. The real Residues (± 1) result from the product-modulus $\pi\pi' = p$, so that in this case in the 4-tic residual relations (7), (8) p may be written instead of π.

4. *Quartic Reciprocity.* Père Pépin has developed† the law of 4-tic reciprocity in the following forms, which are convenient for numerical calculations along with the known partition $p = a^2 + b^2$, viz.

$$(q/\pi)_4 = \frac{(a+b\iota)^k}{(a-b\iota)^k} \pmod{q}, \text{ when } q = 4k+1 \ldots\ldots\ldots (9a),$$

$$(\bar{q}/\pi)_4 \dagger = \frac{(a-b\iota)^k}{(a+b\iota)^k} \pmod{q\dagger}, \text{ when } q = 4k-1 \ldots\ldots\ldots (9b),$$

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* The Table A (at end of this Memoir) shows the value of $(q/\pi)_4$ with $\pi = a + b\iota$

† *Pépin.* Art. 33, Theorem 1. Père Pépin makes but little use of the imaginary roots and residues (ι), but introduces an auxiliary base (l) whereby the residues are all real. But this really complicates his work a good deal. The slight use here made of the imaginary residues (ι) and complex factors ($a \pm b\iota$) makes the work really much shorter and simpler.

Expanding $(a \pm b\iota)^k$ by the binomial theorem, and remembering that $\iota^2 = -1$, it will be found that—

$$(a+b\iota)^k = A + B\iota, \quad (a-b\iota)^k = A - B\iota \text{.....................}(10),$$

where

$$A = a^k - \binom{k}{2} a^{k-2} b^2 + \binom{k}{4} a^{k-4} b^4 - \binom{k}{6} a^{k-6} b^6 + \binom{k}{8} a^{k-8} b^8 - \&c. \text{.....}(11a),$$

$$B = \binom{k}{1} a^{k-1} b - \binom{k}{3} a^{k-3} b^3 + \binom{k}{5} a^{k-5} b^5 - \binom{k}{7} a^{k-7} b^7 + \&c. \text{........}(11b),$$

wherein the general terms are

In A; $\binom{k}{r} a^{k-r} b^r$ [r *even*], with signs alternately $\pm$ ....(12a);

In B; $\binom{k}{r} a^{k-r} b^r$ [r *odd*], with signs alternately $\pm$ ....(12b);

and the series evidently both terminate when a disappears: the series may also be arranged with ascending powers of a, and descending of b.

And, since $(q/\pi)_4 \equiv \rho \equiv$ one of $\iota, -1, -\iota, +1 \pmod q$, it follows that

$$\frac{A+B\iota}{A-B\iota} = \rho = \text{ one of } \iota, -1, -\iota, +1 \pmod q, \text{ when } q = 4k+1 \text{... } (13a),$$

$$\frac{A-B\iota}{A+B\iota} = \rho = \text{ one of } \iota, -1, -\iota, +1 \pmod {q^*}, \text{ when } q = 4k-1 \text{....}(13b).$$

Hence arise the 4 congruences

	$\rho=\iota$	-1	$-\iota$	$+1$	
gives &	$A-B\equiv 0$	$A\equiv 0$	$A+B\equiv 0$	$B\equiv 0$	(mod $q=4k+1$)...(14a),
requires	$A+B\equiv 0$	$A\equiv 0$	$A-B\equiv 0$	$B\equiv 0$	(mod $q=4k-1$*)..(14b).

The solutions of these four congruences for given values of k give the values of the ratios $\mu = a/b$, $\lambda = b/a$, which show the particular Residue ($\rho =$ one of $\iota, -1, -\iota, +1$) of $(\pm q/\pi)_4$, [q, p given]: here μ, λ are the *roots* of the congruences.

The cases $(\pm q/\pi)_4$ are connected by the relation

$$(q/\pi)_4 \cdot (\bar{q}/\pi)_4{}^* = (\bar{1}/\pi)^4 = +1, \text{ when } p = 4\varepsilon + 1,$$
$$= -1, \text{ when } p = 4\omega + 1 \text{......... ....}(15),$$

so that either may be at once inferred from the other.

5. *Reduced Congruences.* The coefficients $\binom{k}{r}$ of the terms in **A**, **B**, and also those of the terms in the four Congruences, are—(except those of the first two and last two

* The *negative sign* is to be attached to the Base q *only when in use in these symbols* $(-q/\pi)_4$, $(-q/p)_4$, when $q = 4k-1$: when q is *in use as a modulus*, it is *always* reckoned +.

terms)—usually much greater than the modulus (q). All such coefficients may be replaced by their Residues modulo q. The functions **A**, **B**, and the Congruences (14a, b), with coefficients so reduced, will be styled the *Reduced* **A**, **B**, and *Reduced Congruences.*

Every value of q has in general *its own special set* of Reduced **A**, **B**, and Reduced Congruences: but this is not always so with the *un*reduced **A**, **B**, and *un*reduced congruences. For since **A**, **B** depend directly on $k=\frac{1}{4}(q\mp 1)$ (and on q only through k), it follows that every pair of values of q, such that

$$q=4k-1,\ q'=4k+1 \text{ (both prime) } q-q'=2 \quad [\text{with same } k]$$

have the *same* (unreduced) **A** & **B**, and *same* (unreduced) Congruences..(16).

[Examples; $(q, q')=(3, 5), (11, 13), (59, 61), (71, 73), (107, 109)$, &c.]

6. *Number and Magnitude of the roots* (μ, λ). The four congruences (14a, b) are all of degree k in the ratios $\mu=a/b$, $\lambda=b/a$, and have therefore exactly k roots each, all $<q$, except that one root μ, λ may $=\infty$.

Taking the four congruences together, each of μ, λ have exactly $4k$ $(=q\mp 1)$ different values, one of which is ∞ (in both cases); and the rest (taken positively) are all $<q$. The roots (μ, λ) may also be arranged in pairs, one pair being always 0, ∞; and the rest are of type $\pm r$, *i.e. equal and of opposite sign*, the greatest (excluding ∞) being always $\pm\frac{1}{2}(q-1)$; [except that ± 2 are excluded when $q=5$].

Hence, it arises that each of μ, λ have *all* the $4k$ $(=q\pm 1)$ values below:—

$$q=4k-1 \text{ gives } \mu,\lambda=\infty,0;\ \pm 1,\ \pm 2,\ \pm 3,\ \ldots,\ \pm\tfrac{1}{2}(q-1)\ldots\ldots(17a),$$

$$q=4k+1 \text{ gives } \mu,\lambda=\infty,0;\ \pm 1,\ \pm 2,\ \pm 3,\ \ldots,\ \pm\tfrac{1}{2}(q-1)\ldots\ldots(17b),$$

[*With two exceptions* in the latter case (as shown below)].

When $q=4k+1$, a prime, then $q=\alpha^2+\beta^2$, and *two* cases are excluded from the above, viz. μ not $\equiv\pm\alpha/\beta$, λ not $\equiv\pm\beta/\alpha$ (mod q).

If these were admissible, they would give $\alpha/\beta=\mu=A/B$, $\beta/\alpha=\lambda=B/A$ (mod q), which would involve $q=p$ (since α, β are mutually prime, and A, B are also mutually prime), which is obviously impossible.

The excluded cases for all $q \not> 100$ are—

$q=$	5,	13,	17,	29,	37,	41,	53,	61,	73,	89,	97;
$\pm\mu, \pm\lambda=$	2,	5,	4,	12,	6,	9,	23,	11,	27,	34,	22;

These excluded values are also given by either of the formulæ [equivalent to $\mu \equiv \pm \alpha/\beta$, $\lambda \equiv \pm \beta/\alpha$ (mod q)].

$\pm\mu$, $\pm\lambda$ not $\equiv \eta$, where η is a root of $\eta^2+1\equiv 0$ (mod q)............(18a),

$\pm\mu$, $\pm\lambda$ not $\equiv g^k$, g^{3k} (mod q), where g is a primitive root of q ..(18b),

7. *Products and Sums of pairs of roots* (λ, μ). The k roots (μ, λ) defined by the *ratios* a/b, b/a, may be arranged, for each one of the congruences, in pairs—[excluding the pair ∞, 0, and also one unique root (± 1) when k is *odd*]—so as to satisfy some of the following results—

$\lambda\mu \equiv +1$, $\lambda\lambda' \equiv \pm 1$, $\mu\mu' \equiv \pm 1$ (mod q)...............(19a),

$\lambda \mp \mu \equiv 0$, $\lambda \mp \lambda' \equiv 0$, $\mu \mp \mu' \equiv 0$ (mod q) ....(19b),

and also, in some cases, by taking one (λ or μ) out of each of the associate congruences $(A\equiv 0, B\equiv 0)$, $(A-B\equiv 0, A+B\equiv 0)$, modulo q as set forth in detail in the following scheme.

q	*Congruence*	*Roots*	*Results* (*mod* q)
Any	Anyone	$\lambda, \lambda', \mu, \mu'$	$\lambda\mu \equiv +1$, $\lambda\lambda' \equiv \pm 1$, $\mu\mu' \equiv \pm 1$
Any	$A\equiv 0$	$\lambda, \lambda', \mu, \mu'$	$\lambda+\lambda'\equiv 0$, $\mu+\mu'\equiv 0$
	$B\equiv 0$	$\lambda, \lambda', \mu, \mu'$	$\lambda+\lambda'\equiv 0$, $\mu+\mu'\equiv 0$
$8\kappa\pm 1$	$A\equiv 0$	$\lambda, \lambda', \mu, \mu'$	$\lambda-\mu\equiv 0$, $\lambda'+\mu'\equiv 0$
	$B\equiv 0$	$\lambda, \lambda', \mu, \mu'$	$\lambda-\mu\equiv 0$, $\lambda'+\mu'\equiv 0$
$8\kappa\pm 5$	$A-B\equiv 0$	λ, μ	$\lambda-\mu\equiv 0$
	$A+B\equiv 0$	λ, μ	$\lambda-\mu\equiv 0$
$8\kappa\pm 1$	$A-B\equiv 0$	λ, μ	$\lambda+\mu\equiv 0$
	$A+B\equiv 0$	λ, μ	$\lambda+\mu\equiv 0$
$8\kappa\pm 1$	$A-B\equiv 0$	λ, μ	$\lambda-\mu'\equiv 0$, $\lambda'-\mu\equiv 0$
	$A+B\equiv 0$	λ', μ'	$\lambda+\lambda'\equiv 0$, $\mu+\mu'\equiv 0$
$8\kappa\pm 5$	$A\equiv 0$	λ, μ	$\lambda-\mu'\equiv 0$, $\lambda'-\mu\equiv 0$
	$B\equiv 0$	λ', μ'	$\lambda+\mu'\equiv 0$, $\lambda'+\mu\equiv 0$
$8\kappa\pm 5$	$A-B\equiv 0$	λ, μ	$\lambda+\lambda'\equiv 0$, $\mu+\mu'\equiv 0$
	$A+B\equiv 0$	λ', μ'	$\lambda+\mu'\equiv 0$. $\lambda'+\mu\equiv 0$
Any	A & $B\equiv 0$	λ, μ	$\lambda \not\equiv \pm\lambda'$ or $\pm\mu'$
	$A\mp B\equiv 0$	λ', μ'	$\mu \not\equiv \pm\lambda'$ or $\pm\mu'$ ....(20).

8. *Sums of roots* (μ, λ). The Residues modulo q of the whole set of k roots (μ or λ) of any one Congruence (14a, b) obey very simple* Rules. [The root μ or $\lambda=\infty$ is here excluded from the summation].

When $(q/\pi)_4 = \mp 1$, then $\Sigma(\mu)\equiv 0$, $\Sigma(\lambda)\equiv 0$ (mod q), *always*........(21a).

When $(q/\pi)_4 = \pm\iota$, then $\Sigma(\mu)$ and $\Sigma(\lambda) = \pm k$ (mod q), [as below]..(21b).

[Use $(-q/\pi)_4$ when $q=4k-1$].

q	k	$(\pm q/\pi)_4=+\iota$ $\Sigma(\mu)$, $\Sigma(\lambda)$	$(\pm q/p)_4=-\iota$ $\Sigma(\mu)$, $\Sigma(\lambda)$	q	k	$(\pm q/\pi)_4=+\iota$ $\Sigma(\mu)$, $\Sigma(\lambda)$	$(\pm q/p)_4=-\iota$ $\Sigma(\mu)$, $\Sigma(\lambda)$	
$8\kappa+1$	ε	$+k$, $-k$	$-k$, $+k$	$4\omega+1$	ω	$+k$, $+k$	$-k$, $-k$	... (21c),
$8\kappa-1$	ε	$-k$, $+k$	$+k$, $-k$	$4\omega-1$	ω	$-k$, $-k$	$+k$, $+k$	... (21d).

* These simple Results serve as easy Tests of the correctness of the Tables.

9. *Products of roots* (μ, λ). The Residues (modulo q) of the continued products of the whole set of k roots (μ or λ) of any one Congruence ($14a$, b) obey very simple* Rules. [The roots μ or $\lambda = 0, \infty$, are here excluded from the continued products].

When $(q/\pi)_4 = \pm\iota$, then $\Pi(\mu)$ & $\Pi(\lambda) \equiv \pm 1$ (mod q), *always*..............($22a$).

When $(q/p)_4 = \mp 1$, then $\Pi(\mu)$ & $\Pi(\lambda) \equiv \pm 1, \pm 4, \pm k$ (mod q) [as below].(22)b.

[Use $-q$ in $(q/\pi)_4$ and $(q/p)_4$ when $q = 4k - 1$].

q	k	$(\pm q/\pi)_4 = +\iota$ $\Pi(\mu)$,	$\Pi(\lambda)$	$(\pm q/\pi)_4 = -\iota$ $\Pi(\mu)$,	$\Pi(\lambda)$	$(\pm q/p)_4 = -1$ $\Pi(\mu)$,	$\Pi(\lambda)$	$(\pm q/p)_4 = +1$ $\Pi(\mu)$,	$\Pi(\lambda)$	
$16\kappa \pm 1$	4κ	$+1$,	$+1$	$+1$,	$+1$	$+1$,	$+1$	-1,	-1	....($22c$),
$8\omega \pm 1$	2ω	-1,	-1	-1,	-1	-1,	-1	$+1$,	$+1$	...($22d$),
$16\kappa \pm 5$	$4\kappa \pm 1$	$+1$,	$+1$	-1,	-1	$\pm k$,	-4	-4,	$\pm k$	....($22e$),
$8\omega \pm 5$	$2\omega \pm 1$	-1,	-1	$+1$,	$+1$	$\mp k$,	$+4$	$+4$,	$\mp k$	...($22f$),

10. *Simple Cases.* The forms of **A**, **B** are such as to lead readily to the correct value $\rho = \iota, -1, -\iota, +1$ of $(\pm q/p)_4$, as shown in the scheme below, as depending on some one of the conditions—

One of $a \equiv 0$	$b \equiv 0$	$a \sim b \equiv 0$	$a + b \equiv 0$	(mod q)
Equiv[t] to $\mu = 0$	$\lambda \equiv 0$	$\lambda = \mu = 1$	$\lambda = \mu = -1$	...(23).

q	k	*Examples* (q)	*Condition* (*modulo* q)	*Result*
$16\kappa + 1$	4κ	17, 97	$ab(a \sim b)(a+b) \equiv 0$	$(q/p)_4 = +1$
$16\kappa - 1$	4κ	31, 47, 79	$ab(a \sim b)(a+b) \equiv 0$	$(\bar{q}/p)_4 = +1$
$8\omega + 1$	2ω	41, 73	$\begin{cases} ab \equiv 0 \\ (a \sim b)(a+b) \equiv 0 \end{cases}$	$(q/p)_4 = +1$ $(q/p)_4 \equiv -1$
$8\omega - 1$	2ω	7, 23, 71	$\begin{cases} ab \equiv 0 \\ (a \sim b)(a+b) \equiv 0 \end{cases}$	$(\bar{q}/p)_4 \equiv +1$ $(\bar{q}/p)_4 \equiv -1$.($23a$)
$8\kappa + 5$	$2\kappa + 1$	5, 13, 29, 37, 53, 61	$\begin{cases} b \equiv 0 \\ a \equiv 0 \end{cases}$	$(q/p)_4 = +1$ $(q/p)_4 = -1$
$16\kappa + 5$	$4\kappa + 1$	5, 37, 53	$\begin{cases} a - b \equiv 0 \\ a + b \equiv 0 \end{cases}$	$(q/\pi)_4 = +\iota$ $(q/\pi)_4 = -\iota$
$8\omega + 5$	$2\omega + 1$	13, 29, 61	$\begin{cases} a - b \equiv 0 \\ a + b \equiv 0 \end{cases}$	$(q/\pi)_4 = +\iota$ $(q/\pi)_4 = -\iota$
$16\kappa + 3$	$4\kappa + 1$	3, 19, 67, 83	$\begin{cases} a - b \equiv 0 \\ a + b \equiv 0 \end{cases}$	$(\bar{q}/\pi)_4 = -\iota$ $(\bar{q}/\pi)_4 = +\iota$
$8\kappa + 3$	$2\omega + 1$	11, 43	$\begin{cases} a - b \equiv 0 \\ a + b \equiv 0 \end{cases}$	$(\bar{q}/\pi)_4 = +\iota$ $(\bar{q}/\pi)_4 = -\iota$.($23b$).

11. *Factors of the Congruences.* The Congruences ($14a$, b), being each of degree k, must evidently break-up into a product of k linear factors of form $(a \mp \mu b)$, $(\lambda a \mp b)$ modulo q, where λ, μ have the values ($< q$) shown in Art. 6.

* These simple Results serve as easy Tests of the correctness of the Tables.

In the scheme below, which shows the factorisation of the Congruences (14a, b) in a general way, P, Q have the following meanings—

P=product of *linear* factors $(\mathrm{a} \mp \mu \mathrm{b})$, $(\lambda \mathrm{a} \mp \mathrm{b})$; [$\mu$, λ all different].

Q=product of *quadratic* factors $(\mathrm{a}^2 - \mu^2 \mathrm{b}^2)$, $(\lambda^2 \mathrm{a}^3 - \mathrm{b}^2)$.; [$\mu$, λ all different].

q	*Factors of* **A, B**	*Factors of* **A∓B**	q	*Factors of* **A, B**	*Factors of* **A∓B**
$16\kappa \pm 1$	$\mathbf{B} \equiv \mathrm{ab}(\mathrm{a}^2 - \mathrm{b}^2).Q$ $\mathbf{A} \equiv Q$	$\mathbf{A} \mp \mathbf{B} \equiv P$	$16\kappa + 5$	$\mathbf{B} \equiv \mathrm{b}Q$ $\mathbf{A} \equiv \mathrm{a}Q$	$\mathbf{A} - \mathbf{B} \equiv (\mathrm{a} - \mathrm{b}).P$ $\mathbf{A} + \mathbf{B} \equiv (\mathrm{a} + \mathrm{b}).P$
$8\omega \pm 1$	$\mathbf{B} \equiv \mathrm{ab}.Q$ $\mathbf{A} \equiv (\mathrm{a}^2 - \mathrm{b}^2).Q$	$\mathbf{A} \mp \mathbf{B} \equiv P$	$8\omega + 5$	$\mathbf{B} \equiv \mathrm{b}Q$ $\mathbf{A} \equiv \mathrm{a}Q$	$\mathbf{A} - \mathbf{B} \equiv (\mathrm{a} + \mathrm{b}).P$ $\mathbf{A} + \mathbf{B} \equiv (\mathrm{a} - \mathrm{b}).P$
			$16\kappa + 3$	$\mathbf{B} \equiv \mathrm{b}Q$ $\mathbf{A} \equiv \mathrm{a}Q$	$\mathbf{A} - \mathbf{B} \equiv (\mathrm{a} + \mathrm{b}).P$ $\mathbf{A} + \mathbf{B} \equiv (\mathrm{a} - \mathrm{b}).P$
			$8\omega + 3$	$\mathbf{B} \equiv \mathrm{b}Q$ $\mathbf{A} \equiv \mathrm{a}Q$	$\mathbf{A} - \mathbf{B} \equiv (\mathrm{a} - \mathrm{b}).P$ $\mathbf{A} + \mathbf{B} \equiv (\mathrm{a} + \mathrm{b}).P$..(24)

12. *Congruence-solutions.* The four Congruences (14a, b) are known (Art. 6) to contain together as their roots (μ, λ) the *whole series* of $(4k - 1)$ *small* numbers $0, \pm 1, \pm 2, \pm 3, \ldots,$ up to $\frac{1}{2}(q-1)$; [*omitting two* when $q = 4k + 1$, see Art. 6].

The Congruences possessing the roots $0, \pm 1$ are known from Art. 10. The factors a, b, a − b, a + b can be divided out of them, thus depressing their degree.

The Congruences possessing the small roots $\pm 2, \pm 3$, &c., are easily found by (numerical) trial. Directly any root $(\pm \mu, \pm \lambda)$ is found, the corresponding factor $(\mathrm{a} \mp \mu \mathrm{b})$, $(\lambda \mathrm{a} \mp \mathrm{b})$ can be divided* out of the congruence to which it belongs, thereby depressing its degree.

The two Congruences $\mathbf{A} \equiv 0$, $\mathbf{B} \equiv 0 \pmod q$ are usually much easier to solve than the other two, in consequence of their roots occurring in pairs $(\pm \lambda, \pm \mu)$, [Art. 7], giving at once the *simple* 2[ic] factors $(\mathrm{a}^2 - \mu^2 \mathrm{b}^2)$, $(\lambda^2 \mathrm{a}^2 - \mathrm{b}^2)$, whereby these two Congruences *can be depressed two degrees* (at one step) for every pair of roots so found.

These Congruences—(when sufficiently depressed by cancellation of factors, as above)—yield congruences of 4th degree in a, b; these will be found to be *trinomial* (involving only a^4, $\mathrm{a}^2\mathrm{b}^2$, b^4) and may often be solved semi-algebraically by modifying their coefficients by aid of the modulus (q), so as to allow of their being resolved into *simple* 2[ic] factors; this process is exhibited in the example below.

When the $2k$ roots (μ, λ) of the above two congruences

* On performing these divisions, there will usually be a remainder (not zero); but, if that remainder be—(as it should be)—a multiple of the modulus q, it may be cancelled out; this shows that the division by the trial factor is *exact*.

$\mathbf{A} \equiv 0$, $\mathbf{B} \equiv 0$ have been found, the remaining k pairs of numbers $(\pm r)$ out of the total of $4k$ numbers of Art. 6, all—(except the pair ∞, 0)—$< \frac{1}{2}(q+1)$, are (see Art. 7) the required $2k$ roots of the two congruences $\mathbf{A}-\mathbf{B}\equiv 0$, $\mathbf{A}+\mathbf{B}\equiv 0$. The *magnitudes* of these roots are hereby known: the proper $(\pm)$ sign to be attached to each can be decided by (numerical trial, noting that—(by Art. 7)

$$\pm\mu \text{ in } \mathbf{A}-\mathbf{B}\equiv 0 \text{ involves } \mp\mu \text{ in } \mathbf{A}+\mathbf{B}\equiv 0,$$
$$\pm\lambda \text{ in } \mathbf{A}-\mathbf{B}\equiv 0 \text{ involves } \mp\lambda \text{ in } \mathbf{A}+\mathbf{B}\equiv 0,$$

so that every root $(\pm r)$ found for one congruence gives the root $(\mp r)$ of the other.

Hereby *all* the roots of the four Congruences (14*a*, *b*) have been found. An example will make this clearer (see Art. 12*a*).

[The finding of all the roots as above—(by direct solution of the Congruences)—is *very laborious*. An easier Method will be explained in Art. 13].

12*a*. *Ex.* Take $q=31$, whence $k=8$, $q=4k-1$.

(1) $\rho=+1$ gives $\mathbf{B}=8a^7b-56a^5b^3+56a^3b^5-8ab^7\equiv 0 \pmod q$
$$=8ab(a^2-b^2)\,Q \pmod q;\quad \mu=\infty,\ 0,\ \pm 1,$$
where $Q=a^4-6a^2b^2+b^4$
$$\equiv a^4-6a^2b^2-216b^4 \pmod q$$
$$\equiv(a^2+12b^2)(a^2-18b^2) \pmod q$$
$$\equiv(a^2-81b^2)(a^2-49b^3) \pmod q;\quad \mu=\pm 9,\ \pm 7.$$

(2) $\rho=-1$ gives $\mathbf{A}=a^8-28a^6b^2+70a^4b^4-28a^2b^6+b^8\equiv 0 \pmod q$
$$\equiv a^8+3a^6b^2+82^4b^4+3a^2b^6+b^8\equiv o \pmod q.$$

On actual trial, it is found that $\mu=\pm 2$, ± 3 satisfy this congruence; whence (by division),

$$\mathbf{A}=(a^2-4b^2)(a^2-9b^2).Q \pmod q,$$
where $Q=a^4+16a^2b^2-6b^4 \pmod Q$
$$\equiv a^4+16a^2b^2-161b^4 \pmod Q$$
$$\equiv(a^2-7b^2)(a^2+23b^2) \pmod Q$$
$$\equiv(a^2-100b^2)(a^2-235b^2) \pmod Q;\quad \mu=\pm 10,\ \pm 15.$$

(3) $\rho=+\iota$ gives $\mathbf{A}+\mathbf{B}\equiv 0 \pmod q$; the Reduced Congruence (Art 5) is
$$a^8+8a^7b+3a^6b^2+6a^5b^3+8a^4b^4-6a^3b^5+3a^2b^6-8ab^7+b^8\equiv 0 \pmod q.$$

(4) $\rho=-\iota$ gives $\mathbf{A}-\mathbf{B}\equiv 0 \pmod q$; the Reduced Congruence is
$$a^8-8a^7b+3a^6b^2-6a^5b^3+8a^4b^4+6a^3b^5+3a^2b^6+8ab^7+b^8\equiv 0 \pmod q.$$

(3, 4). By Art. 12 these last two Congruences contain (as their 16 roots) all the $\pm$integer $<\pm 16$ not already used in the previous Congruences ($\mathbf{A}\equiv 0$, $\mathbf{B}\equiv 0$); in such a way that $\pm\mu$, $\pm\lambda$ in either Congruence involves $\mp\mu$, $\mp\lambda$ in the other. The *magnitudes* of all the roots (μ, λ) being hereby known, the proper $(\pm)$ signs are conveniently determined by (numerical) trial.

In this way it is found that—

$\rho = +\iota$ requires $\mu = 4, \bar{5}, \bar{6}, \bar{8}, \overline{11}, \overline{12}, 13, \overline{14}$.

$\rho = -\iota$ requires $\mu = \bar{4}, 5, 6, 8, 11, 12, \overline{13}, 14$.

13. *Succession-formulæ.** Let the two complete sets of $4k$ roots μ, and $4k$ roots λ be arranged in two arrays, all of four columns and k rows, with subscripts attached indicating their positions in the arrays, as shown below:—where the four columns contain the k roots belonging to the Residues $+\iota, -1, -\iota, +1$, respectively, as shown.

$+\iota$,	-1,	$-\iota$,	$+1$
μ_1,	μ_2,	μ_3,	μ_4
μ_5,	μ_6,	μ_7,	μ_8
μ_9,	μ_{10},	μ_{11},	μ_{12}
μ_{13},	μ_{14},	μ_{15},	μ_{16}
⋮	⋮	⋮	⋮
μ_{4k-3},	μ_{4k-2},	μ_{4k-1},	μ_{4k}

$+\iota$,	-1,	$-\iota$,	$+1$
λ_1,	λ_2,	λ_3,	λ_4
λ_5,	λ_6,	λ_7,	λ_6
λ_9,	λ_{10},	λ_{11},	λ_{12}
λ_{13},	λ_{14},	λ_{15},	λ_{16}
⋮	⋮	⋮	⋮
λ_{4k-3},	λ_{4k-2},	λ_{4k-1},	λ_{4k}

The following *succession-law*† connects every root (μ_r or λ_r) with the next in succession (μ_{r+1} or λ_{r+1}), and thereby enables the whole set of roots (μ or λ) to be computed from a given initial root (μ_1 or λ_1).

$$\mu_{r+1} \equiv \frac{\mu_r\mu_1 - 1}{\mu_r + \mu_1}, \quad \lambda_{r+1} \equiv \frac{\lambda_r + \lambda_1}{1 - \lambda_r\lambda_1} \pmod{q} \quad \text{..................(25)},$$

noting that hereby

$$\mu_2 = (\mu_1^2 - 1) \div 2\mu_1, \quad \lambda_2 = 2\lambda_1 \div (1 - \lambda_1^2), \pmod{q} \text{...........(25a)}.$$

The initial root μ_1 should be a *primitive*‡ root—of the Congruence $\mathrm{A} \mp \mathrm{B} \equiv 0 \pmod{q}$, giving $(q/\pi)_4 \equiv +\iota$ with $\bar{q} = 4k+1$, or of $(\bar{q}/\pi)_4 \equiv +\iota$ with $q = 4k-1$—[*see* 14*a*, *b*]: and the initial root λ_1 should be the reciprocal of μ_1, given by $\lambda_1\mu_1 \equiv +1 \pmod{q}$. With these initial roots and the above formulæ, it will be found that—

Every pair μ_r, λ_r are reciprocal,§ so that $\mu_r\lambda_r \equiv +1 \pmod{q}$ *always*...(26).

More general formulæ, whereby μ_{r+s}, λ_{r+s} may be computed direct from two known roots μ_r, μ_s or λ_r, λ_s, are†

$$\mu_{r+s} = \frac{\mu_r\mu_s - 1}{\mu_r + \mu_s}, \quad \lambda_{r+s} = \frac{\lambda_r + \lambda_s}{1 - \lambda_r\lambda_s} \pmod{q} \text{.................(27)}.$$

* These formulæ are due to Mr. Gosset.

† For formal proof see Art. 13*b*.

‡ The initial root must be a *primitive* root (of the system), otherwise only a limited number of the roots (μ, λ) will be obtained from the succession-formula. In the Table A at end of this Memoir the primitive roots used happen to be the *least* $\pm$ roots ($> \pm 1$) in each case: but this is not always so; *Ex. gr.* $q = -79$ has $\mu = 4$ as the *least root* giving $(-q/\pi)_4 = \iota$, but this is not a primitive root.

§ When either of the sets μ, λ has been computed, the other set (λ, μ) may be computed direct from this reciprocal property more easily than from the succession-formula. Otherwise, if both sets be computed independently, the reciprocal property forms a useful check on the work.

Hence ensue the following *simple** general Results for all values of q—[using $(-q/\pi)_4$ when $q = 4k - 1$].

$\mu_r \equiv$ $\lambda_r \equiv$	$-1,\ 0,\ +1,\ \infty$ $-1,\ \infty,\ +1,\ 0$	$+1,\ 0,\ -1,\ \infty$ ....(28a), $+1,\ \infty,\ -1,\ 0$ ....(28b).
$r =$	$q = 4k-1$ $k,\ 2k,\ 3k,\ 4k$	$q = 4k+1$ $k,\ 2k,\ 3k,\ 4k$

Also $\qquad \mu_{2k-r} \equiv -\mu_{2k+r};\quad \lambda_{2k-r} \equiv -\lambda_{2k+r} \pmod q$..................(29).

These last two Results show that it is *unnecessary to compute more than half* the series of either μ or λ: as the values of μ, and also those of λ, repeat—(but with opposite signs)—at equal distances on either side of the middle point [where $\mu_{2k} \equiv 0,\ \lambda_{2k} \equiv \infty$)].

13*a*. *Computation of a single series.* The succession-formula (27) can be worked so as to yield the roots (μ, λ) of *any one* of the four series $(q/\pi)_4 = +\iota,\ -1,\ -\iota,\ +1$, if desired.

Starting with μ_1, compute μ_2, μ_3, μ_4: these are the *initial roots* of the four series for $+\iota,\ -1,\ -\iota,\ +1$. Then, if μ_σ denote anyone of these initial roots, the roots following in that series are—

$$\mu_{\sigma+4},\ \mu_{\sigma+8},\ \mu_{\sigma+12},\ \ldots,\ \mu_{\sigma+4m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (29).$$

Aso, if *any root* of any one series be μ_s,

The next root to μ_s is μ_{s+4}, (the succession-formula) ....(-9*a*).

13*b*. *Succession-formula—Proof.* Starting with the initial root μ_1, write—(for shortness)—

$$K_1 = \frac{\mu_1 + \iota}{\mu_1 - \iota}, \quad K_r = \frac{\mu_r + \iota}{\mu_r - \iota}, \quad [\text{with } q = 4k+1].$$

Then, by (9*a*)—

$$K_1^{\,k} \equiv +\iota,\ \ K_1^{\,2k} \equiv -1,\ \ K_1^{\,3k} \equiv -\iota,\ \ K_1^{\,4k} \equiv +1 \pmod q,$$
$$K_1^{\,5k} \equiv +\iota, \text{ and so on:—} \pmod q,$$

$$\therefore \qquad \left(\frac{\mu_1 + \iota}{\mu_1 - \iota}\right)^r \equiv K_r \equiv \frac{\mu_r + \iota}{\mu_r - \iota}.$$

Hence $\qquad \dfrac{\mu_{r+s} + \iota}{\mu_{r+s} - \iota} \equiv K_{r+s} \equiv \left(\dfrac{\mu_1 + \iota}{\mu_1 - \iota}\right)^{r+s} \equiv \left(\dfrac{\mu_1 + \iota}{\mu_1 - \iota}\right)^r \cdot \left(\dfrac{\mu_1 + \iota}{\mu_1 - \iota}\right)^s$

$$\equiv K_r . K_s \equiv \frac{\mu_r + \iota}{\mu_r - \iota} \cdot \frac{\mu_s + \iota}{\mu_s - \iota} \pmod q$$

$$\equiv \frac{\mu_r \mu_s + \iota^2 + (\mu_r + \mu_s)\iota}{\mu_r \mu_s + \iota^2 - (\mu_r + \mu_s)\iota} \pmod q,$$

* These simple Results afford useful checks on the work above.

whence $\mu_{r+s} \equiv \dfrac{\mu_r\mu_s - 1}{\mu_r + \mu_s}$ (mod q), [the required formulæ (27)].

Of course this includes formulæ (25), (25a), as particular cases.

13c. *Circular function* analogy.* Writing—

$$\cot\phi_r \equiv \mu_s,\ \ \cot\phi_s \equiv \mu_s,\ \ \tan\phi_r \equiv \lambda_r,\ \ \tan\phi_s \equiv \lambda_s\ \ (\text{mod } q) \ldots\ldots (30).$$

Then $$\cot(\phi_r + \phi_s) \equiv \mu_{r+s},\ \ \tan(\phi_r + \phi_s) \equiv \lambda_{r+s}\ \ (\text{mod } q) \ldots\ldots\ldots\ldots (30a).$$

and Results (28) show that ϕ_{rk} has the following values—

	$q=4k-1$				$q=4k+1$			
$r=$	k	$2k$	$3k$	$4k$	k	$2k$	$3k$	$4k$......(30b),
ϕ_r	$-\frac{1}{4}\pi$,	$-\frac{1}{2}\pi$,	$-\frac{3}{4}\pi$,	$-\pi$	$\frac{1}{4}\pi$,	$\frac{1}{2}\pi$,	$\frac{3}{4}\pi$,	π......(30c),

or values differing from them by a multiple of π.

14. *Form of q with given λ, μ.* It has been shown elsewhere† that a certain linear form of q, say

$$q \equiv R \ (\text{mod } K),\ \text{ or } q = m.K + R;\ [m \text{ may be} -] \ldots\ldots\ldots\ldots (31),$$

gives rise to the same complete set of values of λ, and same complete set of values of μ for all multiples of the modulus K, so that K is the *period of recurrence* of q, as q increases.

The following scheme‡ shows the values of K, R which determine the q-formula for all values of λ, $\mu \not> \pm 4$.

In this Table—whereof the modulus is K—the tabular Residues (R) of form $(4r+3)$ are all marked minus (*i.e.* $\bar{R}$), so as to be virtually of form $(4r+1)$ modulo K, so that *all the tabular values* of $q = m(K+R)$ are virtually of form $q = 4K + 1$. Values of q of form $q' = 4k' + 3$ can be formed by taking negative values of the multiplier m, with the tabular Residues modulo K.

Ex. Take $\mu = 2$; then $K = 40$. The Table gives

	ι	-1	$-\iota$	$+1$
$q = 4k + 1 = 40m +$	37, 33	9, 19	13, 17	1, 29
$q' = 4k' + 3 = -40m +$	-3, -7	9, 19	13, 17	1, -11

* These Results are due to Mr. Gosset.

† See Mr. Gosset's Paper *On the Law of Quartic Reciprocity* in *Messenger of Mathematics*, vol. xli., 1912, pages 76–85.

‡ Part of the Table on pages 84, 85 of the above work. This Table extends to λ and $\mu = \pm 8$. Reference should be made to the original for the construction of this Table.

Writing R_1, R_2, R_3, R_4 for the values of R in the columns of ι, -1, $-\iota$, $+1$, the following relations are seen to obtain—excluding the cases of $\lambda=0$, $\mu=0$.

$\pm\lambda$ have the *same* R_2 set, and the *same* R_4 set...... .....(32*a*),

$\pm\mu$ have the *same* R_2 set, and the *same* R_4 set............(32*b*).

The R_1 sets of $\pm\lambda$ are the *same* as the R_3 sets of $\mp\lambda$.... ......(32*c*).

The R_1 sets of $\pm\mu$ are the *same* as the R_3 sets of $\mp\mu$..........(32*d*).

Each R_1 gives one R_2, R_3, R_4; thus—(neglecting signs)—

$R_2 \equiv R_1^2$, $R_3 \equiv R_1^3$, $R_4 \equiv R_1^4$ (mod K).................(33).

Taking r as the least member of the R_1, R_2, R_3 groups in turn, and taking η_1, η_2, η_3, ..., for the members of the R_4 group: then, noting that $\eta_1 = 1$,

The complete R_1, R_2, or R_3 group is given by the Least + Residues of

$r\eta_1$, $r\eta_2$, $r\eta_3$, ..., (mod K)..........................(34).

Residues (R) *of* q (*mod* K) *for given small values of* λ, μ.

[Use $(-q/\pi)_4$, $(-q/p)_4$ where $q=4k-1$].

☞ Note that $R=4r+1$ (mod K), $q=4k+1$ always (in the Table).

λ μ	K	*Values of R.* $+\iota$	-1	$-\iota$	$+1$
0, .	—	—	—	—	*Always*
., 0	8	—	$\overline{3}$	—	1
1, 1	16	5	$\overline{7}$	$\overline{3}$	1
$\overline{1}$, $\overline{1}$	16	$\overline{3}$	$\overline{7}$	5	1
2, .	20	$\overline{3}$	9	$\overline{7}$	1
$\overline{2}$, .	20	$\overline{7}$	9	$\overline{3}$	1
., 2	40	$\overline{3}$, $\overline{7}$	9, $\overline{19}$	13, 17	1, $\overline{11}$
., $\overline{2}$	40	13, 17	9, $\overline{19}$	$\overline{3}$, $\overline{7}$	1, $\overline{11}$
3, .	80	$\overline{11}$, $\overline{19}$, $\overline{23}$, 33	13, $\overline{31}$, 37, $\overline{39}$	$\overline{7}$, 17, $\overline{21}$, 29	1, $\overline{3}$, 9, $\overline{27}$
$\overline{3}$, .	80	$\overline{7}$, 17, 21, 29	13, $\overline{31}$, 37, $\overline{39}$	11, $\overline{19}$, $\overline{23}$, 33	1, $\overline{3}$, 9, $\overline{27}$
., 3	80	$\overline{7}$, $\overline{11}$, 17, $\overline{19}$	$\overline{3}$, $\overline{27}$, $\overline{31}$, $\overline{39}$	21, $\overline{23}$, 29, 33	1, 9, 13, 37
., $\overline{3}$	80	21, $\overline{23}$, 29, 33	$\overline{3}$, $\overline{27}$, $\overline{31}$, $\overline{39}$	$\overline{7}$, $\overline{11}$, 17, $\overline{19}$	1, 9, 13, 37
4, .	68	$\overline{7}$, $\overline{11}$, $\overline{23}$, 27	9, $\overline{15}$, $\overline{19}$, 25	$\overline{3}$, 5, 29, $\overline{31}$	1, 13, 21, 33
$\overline{4}$, .	68	$\overline{3}$, $\overline{5}$, 29, $\overline{31}$	9, $\overline{15}$, $\overline{19}$, 25	$\overline{7}$, $\overline{11}$, $\overline{23}$, $\overline{27}$	1, 13, 21, 33
., 4	136	$\overline{11}$, $\overline{27}$, $\overline{31}$, 39	9, $\overline{13}$, $\overline{15}$, 21	3, 5, 7, 23	1, $\overline{19}$, 33, $\overline{43}$
		45, 61, $\overline{63}$, $\overline{65}$	25, $\overline{35}$, 49, $\overline{67}$	29, 37, 41, 57	47, 53, 55, 59
., $\overline{4}$	136	3, 5, $\overline{7}$, $\overline{23}$	9, 13, $\overline{15}$, 21	11, 27, 31, 39	1, $\overline{19}$, 33, $\overline{43}$
		29, 37, 41, 57	25, $\overline{35}$, 49, $\overline{67}$	45, 61, 63, 65	$\overline{47}$, 53, $\overline{55}$, $\overline{59}$

15. *Table of 4-tic Residue Criteria.* The Table A (at end of this Memoir) gives the least (±) values of both μ, λ (mod q), which determine whether

$$(q/\pi)_4 = \pm\iota,\quad (q/p)_4 = \pm 1,$$

for *all* small prime Bases $q < 50$, where

$$\rho = 4\varpi + 1,\ \text{a prime},\ = \mathrm{a}^2 + \mathrm{b}^2;\ [a\ odd,\ b\ even].$$

$$\pi = \mathrm{a} + \mathrm{b}\iota.$$

For each prime (q) there are tabulated $k = \frac{1}{4}(\mp 1)$ values of each of μ, λ in the columns headed ι, -1, $-\iota$, $+1$, arranged in the order of the "arrays" shown in Art. 13; these are the values of μ, λ which give the required residuacity-characters (ι, -1, $-\iota$, $+1$) shown at head of the column. Reciprocal values of μ, λ [*i.e.* such that* $\lambda\mu \equiv +1$ (mod q)] are always placed *side by side* for ready recognition.

The Table gives the values of both μ, λ in detail for each of $+q$, $-q$ *separately*, and also for $\pm q$ *together*, for the four small prime Bases $q = 3, 5, 7, 11$ (on account of their relatively greater importance). For the larger prime Bases ($q > 11$), the detail of μ, λ is given only for the simpler of the two cases $\pm q$, viz.

For $+q$; when $q = 4k+1$; [*Ex.* $q = 13, 17, 29, 37, 41$].

For $-q$; when $q = 4k-1$; [*Ex.* $q = 19, 23, 31, 43, 47$].

Either case ($\pm q$) can be inferred from the other by the simple Rule (15). Short directions in the Table itself also make the change from ($\pm q$) to ($\mp q$) quite easy at sight.

15*a*. *Base* $q = \pm 2$. The residuacity-characters (ι, -1, $-\iota$, $+1$) of the Base $q = \pm 2$ depend only on the value of b, (not on that of the ratio b/a). The values of b giving these characters for Base 2 are shown in a small Table at end of the Table A.

15*b*. *Use of Table A* [*q prime*]. To determine whether for a given *prime* modulus $p = 4\varpi + 1$, the Residuacity of a given small *prime* Base ($q < 50$) is

$$(\mp q/\pi)_4 = +\iota,\ \overline{1},\ -\iota,\ +1,\ \text{or}\ (\mp q/p)_4 = -1,\ +1.$$

The 2[ic] partition $p = \mathrm{a}^2 + \mathrm{b}^2$ is supposed† *known*.

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* This property often enables a misprint to be recognised.

† The values of a, b in this partition are given for *all* primes of form $p = 4\varpi + 1$ up to $p \ngtr 100000$ in Cunningham's Tables of *Quadratic Partitions*, London, 1904.

Determine the *integer value* of one of μ, λ from the definition—

$$\mu \equiv a/b, \ \lambda \equiv b/a \ (\text{mod } q)\text{—[with } \textit{proper sign} \text{ of a]},$$

and seek that value of μ or λ in the body of the Table of modulus μ. The required Residuacity-character $(+\iota, \bar{1}, -\iota, +1)$ due to that value of μ or λ will be found at the head of the column of μ or λ. For Examples, see Art. 32*a*.

[As to the sign of a, see (4*a*). When the character ∓ 1 alone is being sought, the sign is *immaterial* (as $\pm\mu$, $\pm\lambda$ give the *same Result*); but the proper sign must be used when the characters $\pm\iota$ are under trial].

Chap. II. *Cubic Residuacity.*

16. *Cubic Residuacity.* In order that Result (2)—the 3-bic residual relation—may be possible, it is clearly necessary that—

$$p \text{ must be of form } p = 6\varpi + 1 \ldots\ldots (36),$$

which involves—

$$p = A^2 + 3B^2 \ [A, B \text{ are one } \textit{odd}, \text{ one } \textit{even}, \text{ and mutually prime}]$$
$$= \tfrac{1}{4}(L^2 + 27M^2) \ [L, M \text{ both } \textit{odd}, \text{ or both } \textit{even}; \text{ otherwise mutually prime}] \ldots (37).$$

[A, B, L, M are, when even, reckoned $+$,
and, when of form $(4a+1)$, are reckoned $+$,
but, when of form $(4a-1)$, are reckoned $-$] ....(37*a*).

This also involves that—

p = product of *two* complex factors, say π_1, π_2 of type π, so that

$$\pi_1 = \phi(\rho) = a_0 + a_1\rho + a_2\rho^2, \text{ or } = \tfrac{1}{2}(L + 3M) + 3M\rho \ldots\ldots (38a),$$
$$\pi_2 = \phi(\rho_2) = a_0 + a_2\rho + a_1\rho^2, \text{ or } = \tfrac{1}{2}(L + 3M) + 3M\rho^2 \ldots\ldots (38b),$$

wherein ρ is a *proper* root of $\rho^3 - 1 \equiv 0$ (the other proper root being ρ^2); the roots of which (say ρ_1, ρ_2, ρ_3) are $\rho_1 = \rho$, $\rho_2 = \rho^2$, $\rho_3 = 1$.

The cubic residual relation of the Base q modulo π is

$$q^{\frac{1}{3}(p-1)} \equiv \text{one of the roots } \rho, \rho^2, 1 \ (\text{mod } \pi) \ [\pi \text{ is one of } \pi_1, \pi_2] \ldots (39).$$

The symbol $(q/\pi)_3$ is commonly used as an *abbreviation*; thus

$$(q/\pi)_3 \text{ means Residue of } q^{\frac{1}{3}(p-1)} \text{ modulo } \pi \ldots\ldots (39a).$$

It is immaterial which of the complex factors (π_1, π_2) is used; but it is essential that the *same* factor* (π_1 or π_2)

* In the Table B (at end of this Memoir) the value of $(q/\pi)_3$ is given with $\pi = \pi_1$.

should be used throughout: the effect of changing the factors (π_1, π_2) is to interchange the Residues ρ, ρ^2. The real Residue (1) results from the product-modulus $\pi_1\pi_2=p$, so that—in this case—in the 3-bic residual relations (39) p may be substituted for π.

16a. *Properties of the roots* (ρ) *of* $\rho^3-1=0$. The *three* roots ρ_1, ρ_2, ρ_3 of $\rho^3-1=0$ are such that

$$1+\rho+\rho^2=0,\ \rho_1=\rho=\tfrac{1}{2}\{-1+\sqrt{(-3)}\},\ \rho_2=\rho^2=\tfrac{1}{2}\{-1-\sqrt{(-3)}\},\ \rho_3=1 \ldots (40a),$$

$$(1-\rho)/(1+\rho)=-\sqrt{(-3)},\ (1-\rho^2)/(1+\rho^2)=+\sqrt{(-3)} \ldots\ldots\ldots\ldots (40b).$$

Also, writing $\quad \theta^2-27=0$, or $\theta=+\sqrt{(27)} \ldots\ldots\ldots\ldots\ldots (41a)$,

$$\theta.\frac{1+\rho}{1-\rho}=+3,\quad \theta.\frac{1+\rho^2}{1-\rho^2}=3 \ldots\ldots\ldots\ldots\ldots (41b),$$

17. *Cubic Reciprocity.* Père Pépin has expressed* the law of 3-bic reciprocity in the following forms, which are convenient for numerical calculations along with the known partition $p=\frac{1}{4}(L^2+27M^2)$, viz.

$$(q/\pi)_3=\frac{(L+M\theta\iota)^k}{(L-M\theta\iota)^k},\ (\text{mod } q),\ \text{when } q=3k+1 \ldots\ldots\ldots\ldots (42a),$$

$$(q/\pi)_3=\frac{(L-M\theta\iota)^k}{(L+M\theta\iota)^k},\ (\text{mod } q),\ \text{when } q=3k-1 \ldots\ldots\ldots\ldots (42b).$$

Expanding $(L\pm M\theta\iota)^k$ by the binomial theorem, and remembering that $\iota^2=-1$, it will be found that

$$(L+M\theta\iota)^k=\mathbf{L}+\mathbf{M}\theta\iota,\ (L-M\theta\iota)^k=\mathbf{L}-\mathbf{M}\theta\iota \ldots\ldots\ldots\ldots (43),$$

where

$$\mathbf{L}=L^k-\binom{k}{2}L^{k-2}M^2\theta^2+\binom{k}{4}L^{k-4}M^4\theta^4-\binom{k}{6}L^{k-6}M^6\theta^6+\binom{k}{8}L^{k-8}M^8\theta^8-\&c. \ldots (43a).$$

$$\mathbf{M}\theta=\binom{k}{1}L^{k-1}M\theta-\binom{k}{3}L^{k-3}M^3\theta^3+\binom{k}{5}L^{k-5}M^5\theta^5-\binom{k}{7}L^{k-7}M^7\theta^7+\&c. \ldots (43b).$$

wherein the general terms are

In $\mathbf{L}$; $\binom{k}{r}L^{k-r}M^r\theta^r$, [$r$ even], with signs alternately $\pm$ .... (43c).

In $\mathbf{M}$; $\binom{k}{r}L^{k-r}M^r\theta^r$, [$r$ odd], with signs alternately $\pm$ .... (43d),

and the series evidently both terminate when L disappears; the series may also be arranged with ascending powers of L, and descending of $M\theta$.

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* *Op. cit.* Art. 16, Theorems IV., V., Art. 25, Theorems VI., VII. Pépin makes little use of the imaginary roots and residues ρ, ρ^2, but introduces an auxiliary Base (t), whereby the Residues are all real But this really complicates the work a good deal. The slight use here made of the imaginary residues (ρ, ρ^2) and complex factors (π_1, π_2) makes the work really much shorter and simpler.

And, since $(q/\pi)_3 \equiv$ one of $\rho, \rho^2, 1$,

therefore $\dfrac{L+M\theta\iota}{L-M\theta\iota}$ = one of $\rho, \rho^2, 1 \pmod q$, when $q=3k+1$........(44a),

$\dfrac{L-M\theta\iota}{L+M\theta\iota}$ = one of $\rho, \rho^2, 1 \pmod q$, when $q=3k-1$........(44b).

Hence arise the following *three* congruences, noting that

$$\theta.\frac{1+\rho}{1-\rho}=+3, \quad \theta.\frac{1+\rho^2}{1-\rho^2}=-3,$$

The Residue	ρ	ρ^2	1	
gives and	$L-3M\equiv 0$	$L+3M\equiv 0$	$M\equiv 0$	(mod $q=3k+1$)......(45a),
requires	$L+3M\equiv 0$	$L-3M\equiv 0$	$M\equiv 0$	(mod $q=3k-1$)......(45b).

The solution of these three congruences for given values of k give the values of the ratios $\mu = L/M$, $\lambda = M/L$, which show the particular Residue (one of $\rho, \rho^2, 1$) of $(q/\pi)_3$ [q, p given]: here μ, λ are the *roots* of the congruences.

18. *Reduced Congruences.* The coefficients $\binom{k}{r}$ of the terms in **L**, **M**θ, and also those of the terms in the three Congruences, are—(except those of the first two and last two terms)—usually much greater than the modulus (q). All such coefficients may be replaced by their Residues modulo q. The functions **L**, **M**, and the Congruences (41a, b) so reduced, will be styled the *Reduced* **L**, **M**, and *Reduced Congruences.*

Every value of q has in general *its own special set* of Reduced **L**, **M**, and Reduced Congruences; but this is not always so with the *un*reduced **L**, **M**, and *un*reduced congruences. For, since **L**, **M** depend directly on $k=\frac{1}{3}(q \mp 1)$, (and on q only through k), it follows that

$q=3k-1$, $q'=3k+1$ (both prime), $q-q'=2$, [with same k]

have the *same* (unreduced **L**, **M**, and the *same* (unreduced) Congruences.(46).

[*Examples*; $(q, q')=$

(5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61), (71, 73), (101, 103), &c.

19. *Formal similarity of* 4-*tic* & 3-*bic conditions.* There is a striking similarity between the 4-tic and 3-bic Residuacity conditions and results.

Compare the expressions (9a, b) of the law of 4-tic Reciprocity with (42a, b) for the law of 3-bic Reciprocity, leading to the formally similar symbols

4-tic symbols; $A=\Phi(a, b\iota)$, $B=\Psi(a, b\iota)$.........[*see* (11a, b)]...(47a),

3-bic symbols; $L=\Phi(L, M\theta\iota)$, $M\theta=\Psi(L, M\theta\iota)$....[*see* (43$a$, b)]...(47b),

and to the similar sets of four Congruences (14a, b) and three Congruences (45a, b) the roots (μ, λ) of which determine the character (ι, $\bar{1}$, $-\iota$, $+1$) of 4-tic and (ρ, ρ^2, 1) of 3-bic residuacity.

20. *Number and magnitude of the roots* (μ, λ). The three congruences (45a, b) are all of degree k in the ratios $\mu = L/M$, $\lambda = M/L$, and have therefore exactly k roots each, all $< q$, except that one root (μ, λ) may $= \infty$.

Taking the three congruences together, each of μ, λ have exactly $3k (= q \mp 1)$ different values, one of which is ∞ (in both cases); and the rest (taken positively) are all $< q$. The roots (μ, λ) may also be arranged in pairs, one pair being always 0, ∞; and the rest are of type $\pm r$, *i.e.* equal and of opposite sign, the greatest (excluding ∞) being always $\pm \frac{1}{2}(q-1)$.

Hence it arises that each of μ, λ have *all* the $3k (= q \mp 1)$ values below :—

$$q = 3k-1 \text{ gives } \mu, \lambda = \infty, 0;\ \pm 1, \pm 2, \pm 3, \ldots, \pm \tfrac{1}{2}(q-1) \ldots (48a),$$

$$q = 3k+1 \text{ gives } \mu, \lambda = \infty, 0;\ \pm 1, \pm 2, \pm 3, \ldots, \pm \tfrac{1}{2}(q-1) \ldots (48b),$$

[with two exceptions in the latter case].

When $q = 3k+1$, a *prime*, then $q = \frac{1}{4}(l^2 + 27m^2)$, and *two* cases are excluded from the above, viz. μ not $\equiv \pm l/m$, λ not $\equiv \pm m/l \pmod q$. If these values were admissible they would give $l/m \equiv \mu = L/M$, $m/l \equiv \lambda = M/L \pmod q$, which would involve $q = p$, which is obviously inadmissible.

The excluded cases for all $q \not> 100$ are

$q =$	7,	13,	19,	31,	37,	43,	61,	67,	73,	79,	97;	($3k+1$).
$\pm\mu =$	1,	5,	7,	2,	11,	4,	20,	24,	22,	17,	19;	l/m.
$\pm\lambda =$	1,	5,	8,	15,	10,	11,	3,	14,	10,	14,	46;	m/l.

21. *Products, and Sums, of pairs of roots.* The k roots (μ, λ), defined by the ratios $\mu = L/M$, $\lambda = M/L$, may be arranged in pairs, k being always even (excluding the pair ∞, 0) to satisfy the results shown in following scheme.

q	*Congruence*	*Roots*	*Results (mod q)*
Any	*Any one*	λ, μ	$\lambda\mu \equiv +1$
Any	$\mathbf{M} \equiv 0$	λ, λ', μ, μ'	$\lambda + \lambda \equiv 0$, $\mu + \mu' \equiv 0$
Any	$\mathbf{L} - 3\mathbf{M} \equiv 0$ $\mathbf{L} + 3\mathbf{M} \equiv 0$	λ, μ λ', μ'	$\lambda + \lambda' \equiv 0$, $\mu + \mu' \equiv 0$
Any	$\mathbf{M} \equiv 0$ $\mathbf{L} \mp 3\mathbf{M} \equiv 0$	λ, μ λ', μ'	λ not $\equiv \pm\lambda'$, μ not $\equiv \pm\mu'$ ... (49).

22. *Simple Cases.* From the want of symmetry in the Congruences (45*a*, *b*) very few general solutions can be recognised at sight. Only two in fact are easy, viz.

For every q; $LM \equiv 0 \pmod{q}$ gives $(q/p)_3 = 1$ ....(50).

[Contrast the numerous simple cases under 4-tic residuacity, Art. 10].

23. *Factors of the Congruences.* The Congruences (45*a*, *b*), being of degree k, must break up into a product of k linear factors of form $(L \mp \mu M)$, $(\lambda L \mp M)$ modulo q, where λ, μ have the values $(<q)$ shown in Art. 20.

In the Result (51) below, which shows the factorisation of the Congruences in a general way, P, Q have the following meanings—

P = product of linear factors $(L \mp \mu M)$, $(\lambda L \mp M)$; [μ, λ all different].

Q = product of quadratic factors $(L^2 \mp \mu^2 M^2)$, $(\lambda^2 L^2 \mp M^2)$; [μ, λ all different].

Then—

For every q; $\mathbf{M} \equiv LM.Q$, $\mathbf{L} \mp 3\mathbf{M} \equiv P$..................(51).

[Contrast the numerous forms under 4-tic residuacity, Art. 9].

24. *Sums of roots* (μ, λ). The Residues modulo q of the whole set of k roots μ, and to a lesser extent of the whole set of k roots λ, of any one Congruence (45*a*, *b*) obey some very simple Rules:—[The root μ or $\lambda = \infty$ is excluded from the summation].

$(q/p)_3 = 1$ gives $\Sigma(\mu) \equiv 0$, $\Sigma(\lambda) \equiv 0 \pmod{q}$, *always* ....(52*a*),

$(q/\pi)_3 = \rho$ gives $\Sigma\mu \equiv -1 \pmod{q}$, $(q/\pi)_3 = \rho^2$ gives $\Sigma(\mu) = +1$...(52*b*).

When $(q/\pi)_3 = \rho$ or ρ^2, the presence of the factor 3 in the two Congruences $\mathbf{L} \mp 3\mathbf{M} \equiv 0$ causes the Residues of $\Sigma(\lambda)$ to be *irregular*.

25. *Products of roots* (μ, λ). The Residues modulo q of the whole set of k roots follow the simple Rules shown in the scheme below:—

q	$(q/\pi)_3 = \rho$ or ρ^2 $\Pi(\mu)$, $\Pi(\lambda)$		$(q/p)_3 = 1$ $\Pi(\mu)$, $\Pi(\lambda)$		
$6k-1$	3,	1/3	$-1/9$,	-9	.........(53*a*),
$6k+1$	1,	1	$-1/27$,	-27	.........(53*b*).

26. *Congruence-solutions.* The three Congruences (45*a*, *b*) are known (Art. 20) to contain together as their roots

$(\pm\mu, \ \pm\lambda)$ the *whole series* of $(3k-1)$ *small numbers* $0, \pm 1, \pm 2, \pm 3, \ldots,$ up to $\frac{1}{2}(q-1)$:—[*omitting two* when $q = 3k+1$, see Art. 20].

The procedure for solution is much the same as that given for the 4-tic case (Art. 12), so need not be detailed here. The Congruence $\mathbf{M} \equiv 0 \pmod q$ is usually much easier to solve than the other two, in consequence of two of its roots being $0, \infty$ (giving at once the factors L, M), and of the other roots occurring always in pairs $(\pm\mu, \pm\lambda)$, which give 2^ic factors (as in Art. 12).

An example will make the procedure clear (see Art. 26*a*).

26*a*. *Ex.* Take $q = 19$; whence $k = 6$, $q = 3k+1$.

(1). $\rho_3 = 1$ gives $\mathbf{M} = 6L^5M - 20L^3M^3.27 + 6LM^5.27^2$

$$\equiv 6LM(L^4 + 5L^2M^2 + 7M^4) \pmod q$$
$$\equiv 6LM.Q \equiv 0 \pmod q;$$
$$Q \equiv L^4 + 5L^2M^2 - 50M^4 \pmod q$$
$$\equiv (L^2 - 5M^2)(L^2 + 10M^2) \pmod q$$
$$\equiv (L^2 - 81M^2)(L^2 - 9M^2) \pmod q;$$
$$\mathbf{M} \equiv 0 \pmod q \text{ gives } \mu = 0, \infty, \ \pm 9, \ \pm 3.$$

(2) & (3). $\rho_2 = \rho$, and $\rho_2 = \rho^2$ give $\mathbf{L} \mp 3\mathbf{M} \equiv 0 \pmod q$;

$$\mathbf{L} \mp 3\mathbf{M} = (L^6 - 15L^4M^2.27 + 15L^2M^4.27^2 - M^6.27^3)$$
$$\mp 3(6L^5M - 2\ L^3M^3.27 + 6LM^5.27^2)$$
$$= (L^6 - 6L^4M^2 + 10L^2M^4 + M^6) \mp (-L^5M - 5L^3M^3 - 7LM^5) \equiv 0 \pmod q.$$

By Art. 20 these two Congruences contain as their roots all the 12 $(\pm)$ integers < 10, not already used for the previous Congruence $(\mathbf{M} \equiv 0)$ [viz $0, \pm 3, \pm 9$], excluding also $\mu = \pm 7$, $\lambda = \pm 8$ (see Art. 20); and in such a way that $\pm\mu$, $\pm\lambda$ in either Congruence involves $\mp\mu$, $\mp\lambda$ in the other (Art. 21). The magnitudes of all the roots (μ, λ) being thus known, the proper $(\mp)$ sign can be conveniently determined by (numerical) trial. In this way it is found that—

	Roots for $\rho_1 = \rho$						Roots for $\rho = {}_2\rho^2$					
$\mu =$	1,	2,	$\bar{4}$,	$\bar{5}$,	$\bar{6}$,	$\bar{8}$	$\bar{1}$,	$\bar{2}$,	4,	5,	6,	8
$\lambda =$	1,	$\bar{9}$,	$\bar{5}$,	$\bar{4}$,	3,	7	$\bar{1}$,	9,	5,	4,	$\bar{3}$,	$\bar{7}$

27. *Successio-formulæ.** Let the two complete sets of $3k$ roots μ, and $3k$ roots λ be arranged in two arrays, each of 3 columns and k rows, with subscripts attached indicating their positions in the arrays, as shown below:—where the

* These formulæ are to due Mr. Gosset.

three columns contain the k roots belonging to the Residues ρ, ρ^2, 1, respectively, as shown.

ρ	ρ^2	1
μ_1,	μ_2,	μ_3
μ_4,	μ_5,	μ_6
μ_7,	μ_8,	μ_9
μ_{10},	μ_{11},	μ_{12}
$\vdots$	$\vdots$	$\vdots$
μ_{3k-2},	μ_{3k-1},	μ_{3k}

ρ	ρ^2	1
λ_1,	λ_2,	λ_3
λ_4,	λ_5,	λ_6
λ_7,	λ_8,	λ_9
λ_{10},	λ_{11},	λ_{12}
$\vdots$	$\vdots$	$\vdots$
λ_{3k-2},	λ_{3k-1},	λ_{3k}

The following succession-law* connects every root (μ_r or λ_r) with the next in succession (μ_{r+1} or λ_{r+1}), and thereby enables the whole set of roots (μ, λ) to be computed from a given initial root (μ_1 or λ_1)

$$\mu_{r+1} \equiv \frac{\mu_r\mu_1 - 27}{\mu_r + \mu_1}, \quad \lambda_{r+1} \equiv \frac{\lambda_r + \lambda_1}{1 - 27\lambda_r\lambda_1}, \quad (\text{mod } q) \ldots\ldots\ldots\ldots\ldots(54),$$

noting that hereby

$$\mu_2 \equiv (\mu_1^2 - 27) \div 2\mu_1, \quad \lambda_2 \equiv 2\lambda_1 \div (1 - 27\lambda_1^2), \quad (\text{mod } q) \ldots\ldots\ldots(54a).$$

The initial root (μ_1) chosen is in every case the *least primitive root* (μ), *i.e.* the least root (μ) *capable of giving all the roots* (μ) of $(q/\pi)_3 = \rho$.

With these initial roots and the above formulæ, it will be found that—

Every pair μ_r, λ_r are *reciprocal*,† so that $\mu_r\lambda_r \equiv +1$ (mod q) *always*...(55).

More general formulæ, whereby μ_{r+s}, λ_{r+s} may be computed direct from two known roots μ_r, μ_s or λ_r, λ_s, are

$$\mu_{r+s} \equiv \frac{\mu_r\mu_s - 27}{\mu_r + \mu_s}, \quad \lambda_{r+s} \equiv \frac{\lambda_r + \lambda_s}{1 - 27\lambda_r\lambda_s}, \quad (\text{mod } q) \ldots\ldots\ldots\ldots\ldots(56).$$

Hence ensue the following *simple* general Results for all values of the Base q.

	$q=3k-1$						$q=3k+1$						
$r =$	$\frac{1}{2}k$	k	$\frac{3}{2}k$	$2k$	$\frac{5}{2}k$	$3k$	$\frac{1}{2}k$	k	$\frac{3}{2}k$	$2k$	$\frac{5}{2}k$	$3k$	
$\mu_r \equiv$	-9	-3	0	$+3$	$+9$	∞	$+9$	$+3$	0	-3	-9	∞	(mod q)...(57a),
$\lambda_r \equiv$	$-\frac{1}{3}k$	$-k$	∞	$+k$	$+\frac{1}{3}k$	0	$-\frac{1}{3}k$	$-k$	∞	$+k$	$-\frac{1}{3}k$	0	...(57b).

Also $\mu_{\frac{3}{2}k-r} \equiv -\mu_{\frac{3}{2}k+r}$, $\lambda_{\frac{3}{2}k-r} \equiv -\lambda_{\frac{3}{2}k+r}$ (mod q)..............(58),

* For formal proof, see Art. 27c.

† When either set (μ, λ) has been computed, the other set (λ, μ) may be computed direct from the reciprocal property (55) more easily than from the succession-formula. Otherwise if both sets be computed independently, the reciprocal propeıty forms a useful check on the work.

These last two Results show that it is *unnecessary to compute more than half* the series of either μ or λ; as the values of μ, and also those of λ, *repeat*—(but *with opposite signs*)—at equal distances on either side of the middle point (where $\mu_{\frac{3}{2}k} \equiv 0$, or $\lambda_{\frac{3}{2}k} \equiv \infty$ (mod q).

27*a*. *Computation of a single series.* The succession-formula (55) can be worked so as to yield the roots (μ, λ) of *any one* of the three series $(q/\pi)_3 = \rho, \rho^2, 1$, if desired.

Starting with μ_1, compute μ_2, μ_3; these are the *initial roots* of the *three* series for ρ, ρ^2, 1. Then, if μ_σ denote any one of these initial roots, the roots following in that series are—

$$\mu_{\sigma+4},\ \mu_{\sigma+6},\ \mu_{\sigma+9},\ \ldots,\ \mu_{\sigma+3m} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(59a).$$

Also, if any root of any one series be μ_s

The next root to μ_s is μ_{s+3}, (the succession-formula)............(59*b*).

27*b*. *4-tic & 3-bic series of $q = 13$.* The series of 12 consecutive roots μ, and of λ also, for $q = 13$, will be seen to be *identical* in the 4-tic and 3-bic Tables—(see Tab. A, B). This identity seems at first sight extraordinary, as the 4-tic and 3-bic reciprocity-conditions (9*a*, *b*; 42*a*, *b*) have *different indices* $[k = \frac{1}{4}(q-1), \frac{1}{3}(q-1)]$.

The reason is that the two succession-formulæ (27, 56) become identical modulo $q = 13$. This identity cannot occur for any other Base (q).

27*c*. *Succession-formula, Proof.* Starting with the initial root μ_1, write (for shortness)—

$$K_1 = \frac{\mu_1 + \theta\iota}{\mu_1 - \theta\iota},\quad K_r = \frac{\mu_r + \theta\iota}{\mu_r - \theta\iota},\ [\text{when } q = 3k+1].$$

Then—by (42*a*)

$$K_1^k \equiv \rho,\ K_1^{2k} \equiv \rho^2,\ K_1^{3k} \equiv 1,\ K_1^{4k} \equiv \rho, \text{ and so on (mod } q),$$

therefore $$\left(\frac{\mu_1 + \theta\iota}{\mu_1 - \theta\iota}\right)^r \equiv K_r = \frac{\mu_r + \theta\iota}{\mu_r - \theta\iota}.$$

Hence $$\frac{\mu_{r+s} + \theta\iota}{\mu_{r+s} - \theta\iota} \equiv K_{r+s} \equiv \left(\frac{\mu_1 + \theta\iota}{\mu_1 - \theta\iota}\right)^{r+s} \equiv \left(\frac{\mu_1 + \theta\iota}{\mu_1 - \theta\iota}\right)^r \cdot \left(\frac{\mu_1 + \theta\iota}{\mu_1 - \theta\iota}\right)^s$$

$$\equiv K_r \,.\, K_s \equiv \frac{\mu_r + \theta\iota}{\mu_r - \theta\iota} \cdot \frac{\mu_s + \theta\iota}{\mu_s - \theta\iota} \pmod q$$

$$\equiv \frac{\mu_r\mu_s + \theta^2\iota^2 + (\mu_r + \mu_s)\theta\iota}{\mu_r\mu_s + \theta^2\iota^2 - (\mu_r + \mu_s)\theta\iota} \pmod q,$$

whence $$\mu_{r+s} = \frac{\mu_r\mu_s - 27}{\mu_r + \mu_s} \pmod q,\ [\text{the required formula (56)}].$$

Of course this includes formulæ (54), (54*a*) as special cases.

27*d*. *Circular function analogy.* The results (52), connecting μ_{r+s}, λ_{r+s} with μ_r, μ_s, λ_r, λ_s, closely resemble the formulæ connecting $\cot(\phi_r+\phi_s)$, $\tan(\phi_r+\phi_s)$ with $\cot\phi_r$, $\cot\phi_s$, $\tan\phi_r$, $\tan\phi_s$. Writing

$$\cot\phi_r=\mu_r/\theta,\ \ \cot\phi_s=\mu_s/\theta;\ \ \tan\phi_r=\theta\lambda_r,\ \ \tan\phi_s=\theta\lambda_s\ (\text{mod } q).$$

Then $$\cot(\phi_r+\phi_s)=\mu_{r+s}/\theta,\ \ \tan(\phi_r+\phi_s)=\theta\lambda_{r+s}\ (\text{mod } q).$$

[Compare this with Art. 13*c*.]

28. *Form of q with given* λ, μ. It has been found that a certain linear form of q, say

$$q=\pm R\ (\text{mod } K),\ \text{or}\ q=m.K\pm R\ \ldots\ldots\ldots\ldots\ldots\ldots(60),$$

gives rise* to the *same* λ and same μ, viz one value for each of the conditions $(q/\pi)_3=\rho,\ \rho^2,\ 1$. [Compare this with Art. 14].

The scheme below shows the values of K and $\pm R$ which determine the q-formula for all values of μ up to ± 5.

It will be seen that, writing R_1, R_2, R_3 for the values of R—(*neglecting* the sign)—in the ρ, ρ^2, 1 columns,

$\pm\mu$ have the same R_3 set ....(61*a*).

The R_3-sets for $\mu=2$, 4, 8, &c., are the Residues of 2^x (mod k).

The residues of the *successive* powers of certain R_1 modulo K give the complete sets of R_1, R_2, R_3............................ ...(61*b*).

Taking r_1 the initial member of the group R_1 or R_2, and η_1, η_2, η_3, &c., the complete set of R_3, then the complete sets of R_1, R_2 are the Residues of $r_1\eta_1$, $r_1\eta_2$ $r_1\eta_3$, &c. ....(61*c*).

μ, κ	*Value of* $\pm R$		
	ρ	ρ^2	1
1, 7	2	3	1
$\bar{1}$, 7	3	2	1
2, 31	3, 6, 7, 12, 14	5, 9, 10, 11, 13	1, 2, 4, 8, 15
$\bar{2}$, 31	5, 9, 10, 11, 13	3, 6, 7, 12, 14	1, 2, 4, 8, 15
3, 18	5	7	1
$\bar{3}$, 18	7	5	1
4, 43	7, 9, 13, 14, 15, 17, 18	3, 5, 6, 10, 12, 19, 20	1, 2, 4, 8, 11, 16, 21
$\bar{4}$, 43	3, 5, 6, 10, 12, 19, 20	7, 9, 13, 14, 15, 17, 18	1, 2, 4, 8, 11, 16, 21
5, 13	2, 3	4, 6	1, 5
$\bar{5}$, 13	4, 6	2, 3	1, 5

29. *Table of* 3-*bic Residue Criteria.* The Table B (separate) following gives the least ($\pm$) values of both μ, λ (mod q), which determine whether

$$(q/\pi)_3=\rho,\ \rho^2,\ 1,\ \ (q/p)_3=1,$$

for *all* small primes $q<50$, where

$p=3\varpi+1$, a prime, $=\frac{1}{4}(L^2+27M^2)$, [L, M both $=\omega$, or both $=\epsilon$].

$\pi=\phi(\rho)=a_0+a_1\rho+a_2\rho^2=\frac{1}{2}(L+3M)+3M\rho$.

* This property is due to Mr. Gosset.

For each prime (q) there are tabulated $k = \frac{1}{3}(q \mp 1)$ values of each of μ, λ in the columns headed $\rho, \rho^2, 1$ arranged in the order of the "arrays" shown in Art. 27; these are the values of μ, λ which give the required residuacity-character $\rho, \rho^2, 1$ shown at head of the column. Reciprocal values of μ, λ [*i.e.* such that* $\lambda\mu \equiv +1 \pmod q$] are always placed *side by side* for ready recognition.

29*a*. *Base* 2. The residuacity-characters ($\rho, \rho^2, 1$) of the Base 2 are most conveniently expressed in terms of the "2ic parts" A, B—[see (37*a*)]—when that Base alone is concerned: but, when occurring in combination with other (odd) Bases, those characters are required in terms of the ratios M/L, L/M—(as for the other Bases). The results μ, λ are accordingly shown in a small special Table, with the modulus 4, at the foot of Table B.

29*b*. *Use of Table B* [*q prime*]. To determine whether for a given *prime* modulus $p = 3\varpi + 1$, the Residuacity of a given small *prime* Base ($q < 50$) is

$$(q/\pi)_3 = \rho,\ \rho^2,\ 1;\ \text{ or } (q/p)_3 = 1.$$

The 2ic partition $p = \frac{1}{4}(L^2 + 27M^2)$ is supposed† *known*.

Determine the *integer value* of one of μ, λ from the definition

$$\mu \equiv L/M,\ \ \lambda \equiv M/L \pmod q\text{—[with \textit{proper signs of} } L, M],$$

and seek that value of μ or λ in the body of the Table of the modulus p. The required Residuacity character ($\rho, \rho^2, 1$) due to that value of μ or λ will be found at the head of the column of μ or λ.

[As to the sign of L, M, see (37*a*). When the character $(q/p)_3 = 1$ alone is being sought, the sign is *immaterial* (as $\pm\lambda$ or $\pm\mu$ give the *same Result*); but the proper ($\pm$) signs must be used when $(q/\pi)_3 = \rho, \rho^2$ are under trial].

Chap. III. *Composite Bases.*

30. *Extension to Composite Bases* (Q). When Q is *composite*, say

$$Q = q_1 q_2 q_3, \ldots;\ [q_1, q_2, \text{\&c., any primes};\ q_1 \text{ may } = q_2, \text{\&c.}] \ldots\ldots(62),$$

the known formulæ

$$(Q/\pi)_4 = (q_1/\pi)_4 \cdot (q_2/\pi)_4 \cdot (q_3/\pi)_4, \text{\&c.} \ldots\ldots\ldots\ldots(63a),$$

$$(Q/\pi)_3 = (q_1/\pi)_3 \cdot (q_2/\pi)_3 \cdot (q_3/\pi)_3, \text{\&c.} \ldots\ldots\ldots\ldots(63b),$$

* This property often enables a misprint to be recognised.

† The values of L, M in this partition are given for *all* primes of form $p = 3\varpi + 1$, up to $p \not> 100000$ in Cunningham's Tables of *Quadratic Partitions*, London, 1904.

enable the use of the Tables A, B to be extended to *composite* Bases Q, where Q is a continued product of primes q and prime-powers q^κ, whenever the bases q are *all* <50—(the limit of the present Tables).

And note that p may be substituted for π in any of the above symbols when that symbol $(q/\pi)_4$ or $(q/\pi)_3 = \pm 1$.

In all cases of composite Bases (Q), the proper ($\pm$) signs of the component Bases (q)—[in the symbols $(q/\pi)_4$]—and also those of all (odd) 2^{ic} parts, viz. a, L, M, must be affixed—[see Art. 3, 4, 16, 17].

31. *Extension to any Base Q.* The finding of the 4-tic or 3-bic Residuacity-character of any Base (Q) whatever may often be brought within the range of the present Tables by first forming the sum or difference of that Base (Q) and any number of multiples (mp) of the prime modulus (p), and then resolving the result ($mp \mp Q$) into its prime factors, say

$$mp \mp Q = p_1^{\alpha} \,.\, p_2^{\beta} \,.\, p_3^{\gamma}, \text{ \&c.} \qquad \text{(56)}.$$

If this can be done in such a way that *all* the prime Bases (q) are <50, the present Tables will suffice to give the required 4-tic or 3-bic residuacity of Q on applying the proper formula.

32. *Examples* (of use of Tables). In the following Examples the moduli (p) have been taken <1000 in order that the Results $[(q/\pi)_4, (q/\pi)_3, (Q/p)_4, (Q/p)_3]$ might be tested from Jacobi's *Canon Arithmeticus.* These Examples have also been so chosen as to emphasize the necessity of affixing the proper ($\pm$) signs to the Bases q (when $q = 4k - 1$, see Art. 3 footnote), and also to the *odd* "2^{ic} parts" a [when $\mathrm{a} = 4\alpha - 1$, see ($4a$)], and L, M [when $L = 4\alpha - 1$, $M = 4\beta - 1$, see ($31a$)].

33. *Computation of Tables.* The principal Tables A, B—(at end of this Memoir)—were computed by the joint authors independently. The short Tables in Art. 14, 28 are due to Mr. Gosset.

32*a*. *Examples of 4-tic residuacity.*

Take $p = 997 = 4.3.83 + 1 = \overline{31}^2 + 6^2$; [here a is $-$].

Find $(799/p)_4$, $(506/p)_4$, $(911/p)_4$.

Ex. 1°. $Q = 799 = 17.47 = q_1 q_2$.

$$q_1 = 17 \text{ gives } \mu \equiv \frac{\mathrm{a}}{\mathrm{b}} \equiv \frac{-31}{6} \equiv \frac{-48}{6} \equiv -8 \pmod{17};\quad (17/\pi)_4 = +\iota.$$

$$q_2 = 47 \text{ gives } \mu \equiv \frac{\mathrm{a}}{\mathrm{b}} \equiv \frac{-31}{6} \equiv \frac{-78}{6} \equiv -13 \pmod{47};\quad (\overline{17}/\pi)_4 = +\iota.$$

$$(799/p)_4 = (17/\pi)_4 \,.\, (47/\pi)_4 = (+\iota) \,.\, (-\iota) \equiv +1.$$

Ex. 2°. $Q=506=2.11.23=q_1q_2q_3$.

$q_1=2$, with $\mathrm{b}=6=8.1-2$, gives $(2/\pi)_4=-\iota$.

$q_2=11$ gives $\mu\equiv\frac{-31}{6}\equiv\frac{-42}{6}\equiv-7\equiv+4$ (mod 11); $(\overline{11}/\pi)_4\equiv+\iota$.

$q_3=23$ gives $\mu\equiv\frac{-31}{6}\equiv\frac{-54}{6}\equiv-9$ (mod 23); $(\overline{23}/p)_4\equiv-1$.

$(506/p)_4=(2/\pi)_4.(11/\pi)_4.(23/p)_4=(-\iota).(-\iota).(+1)\equiv-1$.

Ex. 3°. $Q=911$ (a prime >50); here $Q\equiv-(p-911)\equiv-86\equiv2.\overline{43}=q_1q_2$ (mod p).

$q_1=2$; gives $(2/\pi)_4\equiv-\iota$ as above.

$q_2=43$ gives $\mu\equiv\frac{-31}{6}=\frac{+12}{6}\equiv+2$ (mod 43); $(\overline{43}/\pi)_4\equiv+\iota$.

$(911/p)_4=(2/\pi)_4.(\overline{43}/\pi)_4=(-\iota).(+\iota)=+1$.

32*b*. *Examples of* 3-*bic residuacity.*

Take $p=991=2.9.5.11+1=\frac{1}{4}(61^2+27.3^2)$; [here M is $-$]. Find $(451/p)_3$, $(962/p)_3$, $(113/p)_3$.

Ex. 1°. $Q=451=11.41=q_1q_2$.

$q_1=11$ gives $\mu\equiv\frac{61}{-3}\equiv\frac{+72}{-3}\equiv-24\equiv-2$ (mod 11); $(11/\pi)_3\equiv+\rho$.

$q_2=41$ gives $\mu\equiv\frac{61}{-3}\equiv\frac{+102}{-3}\equiv-34\equiv+7$ (mod 41); $(41/\pi)_3\equiv\rho^2$.

$(451/p)_3=(11/\pi)_3.(41/\pi)_3=\rho.\rho^2=1$.

Ex. 2°. $Q=962=2.13.37=q_1q_2q_3$.

$q_1=2$ gives $\mu\equiv\frac{61}{-3}=\frac{61-4}{-3}=-19\equiv+1$ (mod 4); $(2/\pi)_3=\rho$.

$q_2=13$ gives $\mu\equiv\frac{61}{-3}=\frac{+48}{-3}=-16\equiv-3$ (mod 13); $(13/\pi)_3=\rho^2$.

$q_3=37$ gives $\mu\equiv\frac{61}{-3}\equiv\frac{+24}{-3}\equiv-8$ (mod 37); $(37/p)_3=1$.

$(962/p)_3\equiv(2/\pi)_3.(13/\pi)_3.(37/p)_3=\rho.\rho^2.1=1$.

Ex. 2°, *bis.* $Q=962\equiv-(p-Q)\equiv-29$ (mod p); $q=29$.

$q=29$ gives $\mu\equiv\frac{61}{-3}\equiv\frac{+90}{-3}\equiv-30\equiv-1$; $(29/p)_3\equiv-1$; $(962/p)_3=1$.

Ex. 3°. $Q=113$, (a prime >50); $p+Q=1104=2^4.3.23=q_1q_2q_3$.

$(2^4/\pi)_3=(2^3/\pi)_3.(2/\pi)_3=1.\rho$ as above.

$q_2=3$ gives $\mu\equiv\frac{61}{-3}\equiv\infty$ (mod 3); $(3/\pi)_3=1$;

$q_3=23$ gives $\mu\equiv\frac{61}{-3}\equiv\frac{+84}{-3}\equiv-28\equiv-5$ (mod 23); $(23/\pi)_3=\rho^2$.

$(113/p)_3=(1104/p)_3=(2^4/\pi)_3.(3/\pi)_3.(23/p)_3=\rho.\rho^2.1=1$.

Criteria of 4-*tic Residuacity of* q (*mod* p). TAB. A.

$(q/\pi)_4 = \pm\iota$, and $(q/p)_4 = \pm 1$ when $\mu \equiv$ a/b, $\lambda \equiv$ b/a (mod q).

["Crosswise" means interchanging the $\pm\iota$ columns, and also the ± 1 columns].

q	k	$(p-1)$	*Values of* $(q/\pi)_4$ & $(q/p)_4$.							
			$+\iota$		-1		$-\iota$		$+1$	
			μ	λ	μ	λ	μ	λ	μ	λ
3	1	4ω	1	1	∞	0	$\overline{1}$	$\overline{1}$	0	∞
$\overline{3}$	1	4ω	$\overline{1}$	$\overline{1}$	0	∞	1	1	∞	0
± 3	1	8ϖ	$\overline{1}$	$\overline{1}$	0	∞	1	1	∞	0
7	2	4ω	$\overline{3}$	2	0	∞	2	$\overline{3}$	$\overline{1}$	$\overline{1}$
			$\overline{2}$	3	∞	0	3	$\overline{2}$	1	1
$\overline{7}$	2	4ω	2	$\overline{3}$	$\overline{1}$	$\overline{1}$	$\overline{3}$	2	0	∞
			3	$\overline{2}$	1	1	$\overline{2}$	3	∞	0
± 7	2	8ϖ	2	$\overline{3}$	$\overline{1}$	$\overline{1}$	$\overline{3}$	2	0	∞
			3	$\overline{2}$	1	1	$\overline{2}$	3	∞	0
11	3	4ω	$\overline{1}$	$\overline{1}$	$\overline{2}$	5	3	4	5	$\overline{2}$
			$\overline{4}$	$\overline{3}$	2	$\overline{5}$	4	3	0	∞
			$\overline{3}$	$\overline{4}$	∞	0	1	1	$\overline{5}$	2
$\overline{11}$	3	4ω	3	4	5	$\overline{2}$	$\overline{1}$	$\overline{1}$	$\overline{2}$	5
			4	3	0	∞	$\overline{4}$	$\overline{3}$	2	$\overline{5}$
			1	1	$\overline{5}$	2	$\overline{3}$	$\overline{4}$	∞	0
± 11	3	8ϖ	3	4	5	$\overline{2}$	$\overline{1}$	$\overline{1}$	$\overline{2}$	5
			4	3	0	∞	$\overline{4}$	$\overline{3}$	2	$\overline{5}$
			1	1	$\overline{5}$	2	$\overline{3}$	$\overline{4}$	∞	0
19	5	4ω	*As for* $q=-19$ *below, crosswise.*							
$\overline{19}$	5	4ω	3	$\overline{6}$	$\overline{5}$	$\overline{4}$	8	$\overline{7}$	9	$\overline{2}$
			$\overline{1}$	$\overline{1}$	$\overline{2}$	9	$\overline{7}$	8	$\overline{4}$	$\overline{5}$
			$\overline{6}$	3	0	∞	6	$\overline{3}$	4	5
			7	$\overline{8}$	2	$\overline{9}$	1	1	$\overline{9}$	2
			$\overline{8}$	7	5	4	$\overline{3}$	6	∞	0
± 19	5	8ϖ	*As for* $q=-19$ *above.*							
23	6	4ω	*As for* $q=-23$ *below, crosswise.*							
$\overline{23}$	6	4ω	$\overline{2}$	11	5	$\overline{9}$	4	6	7	10
			$\overline{3}$	$\overline{8}$	$\overline{1}$	$\overline{1}$	$\overline{8}$	$\overline{3}$	10	7
			6	4	$\overline{9}$	5	11	$\overline{2}$	0	∞
			$\overline{11}$	2	9	$\overline{5}$	$\overline{6}$	$\overline{4}$	$\overline{10}$	$\overline{7}$
			8	3	1	1	3	8	$\overline{7}$	$\overline{10}$
			$\overline{4}$	$\overline{6}$	$\overline{5}$	9	2	$\overline{11}$	∞	0
± 23	6	8ω	*As for* $q=-23$ *above.*							
31	8	4ω	*As for* $q=-31$ *below, crosswise.*							
$\overline{31}$	8	4ω	4	8	$\overline{2}$	15	11	$\overline{14}$	7	9
			$\overline{6}$	5	$\overline{3}$	10	$\overline{13}$	$\overline{12}$	$\overline{1}$	$\overline{1}$
			$\overline{12}$	$\overline{13}$	10	$\overline{3}$	5	$\overline{6}$	9	7
			$\overline{14}$	11	15	$\overline{2}$	8	4	0	∞
			$\overline{8}$	$\overline{4}$	$\overline{15}$	2	14	$\overline{11}$	$\overline{9}$	$\overline{7}$
			$\overline{5}$	6	$\overline{10}$	3	12	13	1	1
			13	12	3	$\overline{10}$	6	$\overline{5}$	$\overline{7}$	$\overline{9}$
			$\overline{11}$	14	2	$\overline{15}$	$\overline{4}$	$\overline{8}$	∞	0
± 31	8	8ϖ	*As for* $q=-31$ *above.*							
43	11	4ω	*As for* $q=-43$ *below, crosswise.*							
$\overline{43}$	11	4ω	2	$\overline{21}$	$\overline{10}$	$\overline{13}$	8	$\overline{16}$	$\overline{20}$	15
			19	$\overline{9}$	12	18	17	$\overline{5}$	4	11
			$\overline{6}$	7	14	$\overline{3}$	$\overline{1}$	$\overline{1}$	$\overline{3}$	14
			7	$\overline{6}$	11	4	$\overline{5}$	17	18	12
			$\overline{9}$	19	15	$\overline{20}$	$\overline{16}$	8	$\overline{13}$	$\overline{10}$
			$\overline{21}$	2	0	∞	21	$\overline{2}$	13	10
			16	$\overline{8}$	$\overline{15}$	20	9	$\overline{19}$	$\overline{18}$	$\overline{12}$
			5	$\overline{17}$	$\overline{11}$	$\overline{4}$	$\overline{7}$	6	3	$\overline{14}$
			1	1	$\overline{14}$	3	6	$\overline{7}$	$\overline{4}$	$\overline{11}$
			$\overline{17}$	5	$\overline{12}$	$\overline{18}$	$\overline{19}$	9	20	$\overline{15}$
			$\overline{8}$	16	10	13	$\overline{2}$	21	∞	0
± 43	11	8ϖ	*As for* $q=-43$ *above.*							
47	12	4ω	*As for* $q=-47$ *below, crosswise.*							
$\overline{47}$	12	4ω	2	$\overline{23}$	$\overline{11}$	17	13	$\overline{18}$	$\overline{14}$	10
			22	15	$\overline{8}$	$\overline{6}$	$\overline{5}$	$\overline{19}$	$\overline{12}$	$\overline{4}$
			$\overline{21}$	$\overline{9}$	$\overline{20}$	7	$\overline{16}$	$\overline{3}$	$\overline{1}$	$\overline{1}$
			$\overline{3}$	$\overline{16}$	7	$\overline{20}$	$\overline{9}$	$\overline{21}$	$\overline{4}$	$\overline{12}$
			$\overline{19}$	$\overline{5}$	$\overline{6}$	$\overline{8}$	15	22	10	$\overline{14}$
			$\overline{18}$	13	17	$\overline{11}$	$\overline{23}$	2	0	∞
			23	$\overline{2}$	$\overline{17}$	11	18	$\overline{13}$	$\overline{10}$	14
			$\overline{15}$	$\overline{22}$	6	8	19	5	4	12
			9	21	$\overline{7}$	20	3	16	1	1
			16	3	20	$\overline{7}$	21	9	12	4
			5	19	8	6	$\overline{22}$	$\overline{15}$	14	$\overline{10}$
			$\overline{13}$	18	11	$\overline{17}$	$\overline{2}$	23	∞	0
± 47	12	8ϖ	*As for* $q=-47$, *above.*							

TAB. A (*continued*).

Criteria of 4-tic Residuacity of q (mod p).

$(q/\pi)_4 = \pm\iota$ *and* $(q/p)_4 = \pm 1$ *when* $\mu \equiv \mathrm{a}/\mathrm{b}$, $\lambda \equiv \mathrm{b}/\mathrm{a}$ (*mod* q).

["Crosswise" means interchanging the $\pm\iota$ columns, and also the ± 1 columns.]

q	k	$(p-1)$	*Values of* $(q/\pi)_4$ & $(q/p)_4$.							
			$+\iota$		-1		$-\iota$		$+1$	
			μ	λ	μ	λ	μ	λ	μ	λ
5	1	4ω	1	1	0	∞	$\bar{1}$	$\bar{1}$	∞	0
$\bar{5}$	1	4ω	$\bar{1}$	$\bar{1}$	∞	0	1	1	0	∞
± 5	1	8ϖ	1	1	0	∞	$\bar{1}$	$\bar{1}$	∞	0
13	3	4ω	$\bar{2}$	6	$\bar{4}$	3	1	1	3	$\bar{4}$
			6	$\bar{2}$	2	∞	$\bar{6}$	2	$\bar{3}$	4
			$\bar{1}$	$\bar{1}$	4	$\bar{3}$	$\bar{2}$	$\bar{6}$	∞	0
$\overline{13}$	3	4ω	*As for $q = +13$ above, crosswise.*							
± 13	3	8ϖ	*As for $q = +13$ above.*							
17	4	4ω	$\bar{2}$	8	$\bar{5}$	$\bar{7}$	6	3	1	1
			3	6	$\bar{7}$	$\bar{5}$	8	$\bar{2}$	0	∞
			$\bar{8}$	2	7	5	$\bar{3}$	$\bar{6}$	$\bar{1}$	$\bar{1}$
			$\bar{6}$	$\bar{3}$	5	7	2	$\bar{8}$	∞	0
$\overline{17}$	4	4ω	*As for $q = +17$ above, crosswise.*							
± 17	4	8ϖ	*As for $q = +17$ above.*							
29	7	4ω	$\bar{3}$	$\overline{10}$	$\overline{11}$	$\bar{8}$	6	5	13	9
			$\bar{4}$	7	$\overline{14}$	2	1	1	2	$\overline{14}$
			7	$\bar{4}$	9	13	5	6	$\bar{8}$	$\overline{11}$
			$\overline{10}$	$\bar{3}$	0	∞	10	3	8	11
			$\bar{5}$	$\bar{6}$	$\bar{9}$	$\overline{13}$	$\bar{7}$	4	$\bar{2}$	14
			$\bar{1}$	$\bar{1}$	14	$\bar{2}$	4	$\bar{7}$	$\overline{13}$	$\bar{9}$
			$\bar{6}$	$\bar{5}$	11	8	3	10	∞	0
$\overline{29}$	7	4ω	*As for $q = +29$ above, crosswise.*							
± 29	7	8ϖ	*As for $q = +29$ above.*							

q	k	$(p-1)$	*Values of* $(q/\pi)_4$ & $(q/p)_4$.							
			$+\iota$		-1		$-\iota$		$+1$	
			μ	λ	μ	λ	μ	λ	μ	λ
37	9	3ω	2	$\overline{18}$	10	$\overline{11}$	17	$\overline{13}$	$\bar{8}$	$\overline{14}$
			9	$\bar{4}$	15	5	$\bar{7}$	$\overline{16}$	3	$\overline{12}$
			1	1	$\overline{12}$	3	$\overline{16}$	$\bar{7}$	5	15
			$\bar{4}$	9	$\overline{14}$	$\bar{8}$	$\overline{13}$	17	$\overline{11}$	10
			$\overline{18}$	2	0	∞	18	$\bar{2}$	11	$\overline{10}$
			13	$\overline{17}$	14	8	4	$\bar{9}$	$\bar{5}$	$\overline{15}$
			16	7	12	$\bar{3}$	$\bar{1}$	$\bar{1}$	$\bar{3}$	12
			7	16	$\overline{15}$	$\bar{5}$	$\bar{9}$	4	8	14
			$\overline{17}$	13	$\overline{10}$	11	$\bar{2}$	18	∞	0
$\overline{37}$	9	4ω	*As for $q = +37$ above, crosswise.*							
± 37	9	8ϖ	*As for $q = +37$ above.*							
41	10	4ω	$\bar{4}$	10	$\bar{7}$	$\bar{6}$	5	$\bar{8}$	20	$\bar{2}$
			18	16	$\overline{14}$	$\bar{3}$	$\overline{19}$	$\overline{13}$	11	15
			17	$\overline{12}$	1	1	$\overline{12}$	17	15	11
			$\overline{13}$	$\overline{19}$	$\bar{3}$	$\overline{14}$	16	18	$\bar{2}$	20
			$\bar{8}$	5	$\bar{6}$	$\bar{7}$	10	$\bar{4}$	0	∞
			$\overline{10}$	4	6	7	8	$\bar{5}$	2	$\overline{20}$
			$\overline{16}$	$\overline{18}$	3	14	13	19	$\overline{15}$	$\overline{11}$
			12	$\overline{17}$	$\bar{1}$	$\bar{1}$	$\overline{17}$	12	$\overline{11}$	$\overline{15}$
			19	13	14	3	$\overline{18}$	$\overline{16}$	$\overline{20}$	2
			$\bar{5}$	8	7	6	4	$\overline{10}$	∞	0
$\overline{41}$	10	4ω	*As for $q = +41$ above, crosswise.*							
± 41	10	8ϖ	*As for $q = +41$ above.*							

q	–	$(p-1)$	$+\iota$	-1	$-\iota$	$+1$
2	–	4ω	$\mathrm{b} = 8\beta + 2$	.	$\mathrm{b} = 8\beta - 2$	.
$\bar{2}$	–	4ω	$\mathrm{b} = 8\beta - 2$	.	$\mathrm{b} = 8\beta + 2$	.
± 2	–	8ϖ	.	$b = 4\omega$	.	$\mathrm{b} = 4\varepsilon$

Tab. B.

Criteria of 3-bic Residuacity of q (mod p).

$(q/\pi)_3 = \rho, \rho^2$; $(q/p)_3 = 1$, when $\mu \equiv L/M$, $\lambda \equiv M/L$ (mod q).

		$(q/\pi)_3$				$(q/p)_3$	
		ρ		ρ^2		1	
q	k	μ	λ	μ	λ	μ	λ
3	–	$\bar{1}$	$\bar{1}$	1	1	∞	0
5	2	1	1	2	$\bar{2}$	0	∞
		$\bar{2}$	2	$\bar{1}$	$\bar{1}$	∞	0
7	2	2	$\bar{3}$	3	$\bar{2}$	0	∞
		$\bar{3}$	2	$\bar{2}$	3	∞	0
11	4	$\bar{1}$	$\bar{1}$	2	$\bar{5}$	4	3
		$\bar{3}$	$\bar{4}$	$\bar{5}$	2	0	∞
		5	$\bar{2}$	3	4	$\bar{4}$	$\bar{3}$
		$\bar{2}$	5	1	1	∞	0
13	4	$\bar{2}$	6	$\bar{4}$	3	1	1
		3	$\bar{4}$	6	$\bar{2}$	0	∞
		$\bar{6}$	2	$\bar{3}$	4	$\bar{1}$	$\bar{1}$
		4	$\bar{3}$	2	$\bar{6}$	∞	0
17	6	$\bar{1}$	$\bar{1}$	$\bar{4}$	4	8	$\bar{2}$
		$\bar{5}$	$\bar{7}$	$\bar{2}$	8	$\bar{3}$	$\bar{6}$
		6	3	7	5	0	∞
		$\bar{7}$	$\bar{5}$	$\bar{6}$	$\bar{3}$	3	6
		2	$\bar{8}$	5	7	$\bar{8}$	2
		4	$\bar{4}$	1	1	∞	0
19	6	2	$\bar{9}$	$\bar{1}$	$\bar{1}$	9	$\bar{2}$
		$\bar{6}$	3	5	4	3	$\bar{6}$
		$\bar{8}$	7	4	5	0	∞
		$\bar{4}$	$\bar{5}$	8	$\bar{7}$	$\bar{3}$	6
		$\bar{5}$	$\bar{4}$	6	$\bar{3}$	$\bar{9}$	2
		1	1	$\bar{2}$	9	∞	0
23	8	1	1	10	7	11	$\bar{2}$
		$\bar{9}$	5	$\bar{7}$	$\bar{10}$	$\bar{2}$	11
		6	4	$\bar{3}$	$\bar{8}$	$\bar{8}$	$\bar{3}$
		5	$\bar{9}$	4	6	0	∞
		$\bar{4}$	$\bar{6}$	$\bar{5}$	9	8	3
		3	8	$\bar{6}$	$\bar{4}$	2	$\bar{11}$
		7	10	9	$\bar{5}$	$\bar{11}$	2
		$\bar{10}$	$\bar{7}$	$\bar{1}$	$\bar{1}$	∞	0
29	10	4	$\bar{7}$	$\bar{5}$	$\bar{6}$	$\bar{11}$	$\bar{8}$
		6	5	$\bar{9}$	$\bar{13}$	1	1
		7	$\bar{4}$	8	11	$\bar{2}$	14
		$\bar{3}$	$\bar{10}$	$\bar{10}$	$\bar{3}$	$\bar{13}$	$\bar{9}$
		12	$\bar{12}$	14	$\bar{2}$	0	∞
		$\bar{14}$	2	$\bar{12}$	12	13	9
		10	3	3	10	2	$\bar{14}$
		$\bar{8}$	$\bar{11}$	$\bar{7}$	4	$\bar{1}$	$\bar{1}$
		9	13	$\bar{6}$	$\bar{5}$	11	8
		5	6	$\bar{4}$	7	∞	0
31	10	$\bar{4}$	$\bar{8}$	13	12	5	$\bar{6}$
		15	$\bar{2}$	9	7	6	$\bar{5}$
		$\bar{10}$	3	$\bar{12}$	$\bar{13}$	$\bar{11}$	14
		3	$\bar{10}$	8	4	$\bar{7}$	$\bar{9}$
		14	$\bar{11}$	1	1	0	∞
		$\bar{1}$	$\bar{1}$	$\bar{14}$	11	7	9
		$\bar{8}$	$\bar{4}$	$\bar{3}$	10	11	$\bar{14}$
		12	13	10	$\bar{3}$	$\bar{6}$	5
		$\bar{9}$	$\bar{7}$	$\bar{15}$	2	$\bar{5}$	6
		$\bar{13}$	$\bar{12}$	4	8	∞	0
37	12	2	$\bar{18}$	$\bar{15}$	$\bar{5}$	$\bar{7}$	$\bar{16}$
		$\bar{14}$	$\bar{8}$	$\bar{17}$	13	9	$\bar{4}$
		16	7	$\bar{10}$	11	$\bar{8}$	$\bar{14}$
		1	1	4	$\bar{9}$	3	$\bar{12}$
		18	$\bar{2}$	6	$\bar{6}$	12	$\bar{3}$
		13	$\bar{17}$	$\bar{5}$	$\bar{15}$	0	∞
		5	15	$\bar{13}$	17	$\bar{12}$	3
		$\bar{6}$	6	$\bar{18}$	2	$\bar{3}$	12
		$\bar{4}$	9	$\bar{1}$	$\bar{1}$	8	14
		10	$\bar{11}$	$\bar{16}$	$\bar{7}$	$\bar{9}$	4
		17	$\bar{13}$	14	8	7	16
		15	5	$\bar{2}$	18	∞	0

TAB. B (*continued*).

Criteria of 3-bic Residuacity of q (mod p).

$(q/\pi)_3 = \rho, \rho^2$; $(q/p)_3 = 1$, *when* $\mu \equiv L/M$, $\lambda \equiv M/L$ (*mod q*).

q	k	$(q/\pi)_3$ ρ μ	λ	ρ^2 μ	λ	$(q/p)_3$ 1 μ	λ
41	14	5	$\bar{8}$	8	$\bar{5}$	1	1
		10	$\bar{4}$	7	6	$\bar{13}$	$\bar{19}$
		$\bar{9}$	9	18	16	17	$\bar{12}$
		$\bar{16}$	$\bar{18}$	6	7	4	$\bar{10}$
		$\bar{19}$	$\bar{13}$	$\bar{3}$	$\bar{14}$	20	$\bar{2}$
		$\bar{2}$	20	15	11	$\bar{14}$	$\bar{3}$
		$\bar{12}$	17	$\bar{11}$	$\bar{15}$	0	∞
		11	15	12	$\bar{17}$	14	3
		$\bar{15}$	$\bar{11}$	2	$\bar{20}$	$\bar{20}$	2
		3	14	19	13	$\bar{4}$	10
		$\bar{6}$	$\bar{7}$	16	18	$\bar{17}$	12
		$\bar{18}$	$\bar{16}$	9	$\bar{9}$	13	19
		$\bar{7}$	$\bar{6}$	$\bar{10}$	4	$\bar{1}$	$\bar{1}$
		$\bar{8}$	5	$\bar{5}$	8	∞	0
43	14	6	$\bar{7}$	$\bar{10}$	$\bar{13}$	11	4
		20	$\bar{15}$	$\bar{8}$	16	16	$\bar{8}$
		9	$\bar{19}$	19	$\bar{9}$	$\bar{12}$	$\bar{18}$
		$\bar{5}$	17	$\bar{14}$	3	$\bar{13}$	$\bar{10}$
		15	$\bar{20}$	3	$\bar{14}$	$\bar{1}$	$\bar{1}$
		2	$\bar{21}$	$\bar{18}$	$\bar{12}$	$\bar{21}$	2
		$\bar{7}$	6	$\bar{17}$	5	0	∞
		17	$\bar{5}$	7	$\bar{6}$	21	$\bar{2}$
		18	12	$\bar{2}$	21	1	1
		$\bar{3}$	14	$\bar{15}$	20	13	10
		14	$\bar{3}$	5	$\bar{17}$	12	18
		$\bar{19}$	9	$\bar{9}$	19	$\bar{16}$	8
		8	$\bar{16}$	$\bar{20}$	15	$\bar{11}$	$\bar{4}$
		10	13	$\bar{6}$	7	∞	0
47	14	7	$\bar{20}$	15	22	$\bar{5}$	$\bar{19}$
		16	3	18	$\bar{13}$	19	5
		$\bar{14}$	10	$\bar{9}$	$\bar{21}$	$\bar{2}$	23
		20	$\bar{7}$	$\bar{8}$	$\bar{6}$	$\bar{11}$	17
		$\bar{21}$	$\bar{9}$	$\bar{1}$	1	$\bar{10}$	$\bar{14}$
		$\bar{3}$	$\bar{16}$	$\bar{12}$	$\bar{4}$	$\bar{6}$	$\bar{8}$
		$\bar{22}$	$\bar{15}$	$\bar{13}$	18	4	12
		$\bar{17}$	11	$\bar{23}$	2	0	∞
		23	$\bar{2}$	17	$\bar{11}$	$\bar{4}$	$\bar{12}$
		13	$\bar{18}$	22	15	6	8
		12	4	3	16	$\bar{10}$	14
		1	1	21	9	11	$\bar{17}$
		8	6	$\bar{20}$	7	2	$\bar{23}$
		9	21	14	$\bar{10}$	$\bar{19}$	$\bar{5}$
		$\bar{18}$	13	$\bar{16}$	$\bar{3}$	5	19
		$\bar{15}$	$\bar{22}$	$\bar{7}$	20	∞	0

p	k	ρ μ λ	ρ^2 μ λ	1
2	—	1, 1	$\bar{1}$, $\bar{1}$	$LM \equiv 0$
	—	mod 4	mod 4	mod 8

Appendix.

Pépin's Results. In respect to 4-tic residuacity Pépin describes a Base q as of *class* 1, 2, 3, 0 modulo p when—in the notation of the present Memoir—$(q/\pi)_4 = +\iota, -1, -\iota, +1$ respectively, and gives the "class-numbers" of the* Bases—

$$q = 5,\ 13,\ 17,\ 29;\quad \overline{3},\ \overline{7},\ \overline{11},\ \overline{19},\ \overline{23};$$

In respect to 3-bic residuacity he describes q as of *class* 1, 2, 0, when—in the notation of the present Memoir—$(q/\pi)_3 = \rho, \rho^2, 1$, where $\rho^3 - 1 = 0$; and gives the "class-numbers" of the† Bases—

$$q = 7,\ 13,\ 19,\ 31;\quad 5,\ 11,\ 17,\ 23,\ 29;$$

Errata in Pépin's Work. In the course of preparing the present Tables, the following Errata has been found in his Text and Results:

Errata in the Text.

Page	*Line*	*For*	*Read*	*Page*	*Line*	*For*	*Read*
10	7 up	Σp^{th}	$\Sigma\rho^{th}$	57	14 up	-5	-3
17	13	a_{12-1}	a_{n-1}	,,	10 up	4	3
22	last	p	ρ	59	1	a^2+2b^2	a^2+b^2
23	13 up	por	par	,,	2	b^2-a^2	b^2-3a^2
,,	2 up	θ^2	θ^{ti}	89	12 up	$-2a_2)$	$-2a_4)$

Errata in Tables.

Page	*Line*	*For*	*Read*	*Page*	*Line*	*For*	*Read*
33	7	$\frac{L}{M}=8, 16$	23, 15	41	3	$\frac{L}{M}=19, 23$	10, 6
,,	8	$\frac{L}{M}=23, 15$	8, 16	,,	3	——	Add $\frac{L}{M}=21$
38	11	$2M, 3M$	$9M, 8M$	,,	4	$\frac{L}{M}=10, 6$	19, 23
,,	23	$9M, 8M$	$2M, 3M$	50	11	$a\equiv 4a$	$b\equiv 4a$
,,	5 up	$M, 7M$	$16M, 10M$	51	6	$b\equiv 6, a$	$b\equiv 6a$
,,	4 up	$16M, 10M$	$M, 7M$				
40	10 & 9 up	Interchange these lines.					

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* *Op. cit.*, pages 50–53, 57–61. † *Op. cit.*, pages 28–33, 36–41.

ON PLANE CURVES OF DEGREE n WITH TANGENTS OF n-POINT CONTACT.

Second Paper.

By *Harold Hilton.*

§ 1. In a former paper* properties of such curves were discussed when the curves were unicursal. Here we consider some other cases.

Suppose an n-ic (curve of degree n) has three *real* n-point tangents (tangents having n-point contact) forming a triangle ABC, the points of contact being D, E, F.

When n is odd, D, E, F must be collinear. When n is even, either D, E, F are collinear, or else AD, BE, CF are concurrent and a conic touches BC, CA, AB at D, E, F. This is proved at once by taking ABC as triangle of reference.

If the real n-point tangents are concurrent, their points of contact must be collinear whether n is even or odd (see §8).

§ 2. Consider first of all the case in which three real n-point tangents of an n-ic have collinear points of contact, n being odd or even.

It readily follows by repetition of the preceding argument that, if three real n-point tangents have collinear points of contact, then the points of contact of all the real n-point tangents are collinear.

Now we have the theorem :—

I. "If

$$l_1x + m_1y + 1 = 0,\ \ l_2x + m_2y + 1 = 0,\ \ \ldots,\ \ l_hx + m_hy + 1 = 0$$

are lines, each meeting an n-ic in r points at infinity, the equation of the n-ic may be put in the form

$$(l_1x + m_1y + 1)\ldots(l_hx + m_hy + 1)\, u_{n-h} + u_{n-r} = 0,$$

where $u_{n-h} = 0$ is some $(n-h)$-ic and $u_{n-r} = 0$ some $(n-r)$-ic".

Consider now the case of an n-ic having n real n-point tangents with their points of contact collinear. Suppose the chord of contact projected to infinity, so that the tangents become the asymptotes†

$$l_1x + m_1y + 1 = 0,\ \ l_2x + m_2y + 1 = 0,\ \ \ldots,\ \ l_nx + m_ny + 1 = 0 \ldots\text{(i)}.$$

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* *Messenger of Mathematics*, vol. xlix., p. 129. This paper will be referred to as "Paper 1".

† If any one of these asymptotes, say the first, passes through the origin, replace $l_1x + m_1y + 1$ by $lx_1 + y$ or $x + m_1y$, etc.

Then the n-ic becomes

$$(l_1x+m_1y+1)\,(l_2x+m_2y+1)\ldots(l_nx+m_ny+1)=k\ldots\text{(ii)},$$

where k is some constant; as is evident on putting $r=h=n$ in Theorem I.

If we suppose the tangents (i) and their infinite chord of contact to be given, we have given n^2 points at infinity on the n-ic; whereas an n-ic is completely determined by $\frac{1}{2}n(n+3)$ points in general. The explanation is contained in the Theorem

II. "Suppose $n-r+1$ tangents of an n-ic have (at least) $(r+1)$-point contact at points which lie on a line meeting the curve again in P. Then if the tangent at P meets the curve at least r times at P, it must have at least $(r+1)$-point contact at P".

For, putting $h=n-r+2$ in Theorem I., we have l_1x+m_1y, ..., $l_{n-r+1}x+m_{n-r+1}y$ all factors of the terms of degree $n-r$ in u_{n-r}. Therefore these terms vanish, and all the $n-r+2$ tangents have $(r+1)$-point contact.

Now it follows from Theorem II. that, if an n-ic has n given tangents of n, n, $n-1$, $n-2$, ..., 2-point contact with the curve at infinity, which is equivalent to

$$n+n+(n-1)+(n-2)+\ldots+2=\tfrac{1}{2}n(n+3)-1$$

conditions, then the n given tangents have all n-point contact. The equation of an n-ic with n such tangents of n-point contact at infinity has therefore one arbitrary constant in its equation, namely, k in equation (ii).

We see that to be given n-point tangents at n given collinear points, and also to be given one more point, determines an n-ic uniquely.

In general, to be given $(n-r+1)$-point tangents at n given collinear points, and also to be given an r-ple point, determines an n-ic.

If the tangents are the lines (i) touching at infinity, and the r-ple point is the origin, the n-ic is

$$p_n+p_{n-1}+p_{n-2}+\ldots+p_r=0,$$

where p_s denotes the sum of the products of l_1x+m_1y, l_2x+m_2y, ... s at a time*.

If $n\,(n-1)$-point tangents at n fixed collinear points are given, and the n-ic has a cusp, this cusp lies on a fixed $2\,(n-2)$-ic. The case $n=3$ is well known.

* An n-ic with $n-r+1$ n-point tangents at infinity and an r-ple point at the origin is

$$(l_1x+m_1y+1)\ldots(l_{n-r+1}x+m_{n-r+1}y+1)\,u_{r-1}=1,$$

the coefficients of u_{r-1} being chosen to make the origin an r-point; and so in § 5.

§ 3. The product of the distances of any point on the n-ic of § 2 (ii) from its asymptotes is constant.

The only real inflexions of the n-ic are the $n(n-2)$ inflexions which coincide with the points of contact of the lines of § 2 (i).

For, if there were another real inflexion, we might suppose it to be the origin and $y=0$ to be inflexional tangent. This would involve $k=1$, $l_1+l_2+\ldots=0$, $l_1l_2+l_1l_3+\ldots=0$, giving $l_1^2+l_2^2+\ldots=0$; which is impossible.

If the tangent at any point P of the n-ic meets the asymptotes in $P_1, P_2, \ldots, P_n$, then

$$1/PP_1+1/PP_2+\ldots+1/PP_n=0.$$

The same holds for any line through a double point P of the n-ic, if such exists.

For, if the tangent (or line) is taken as $y=0$ and P as origin, $k=1$ and $l_1+l_2+\ldots=0$.

Similarly we see that, if the n-ic has a double point, it must be an acnode (with unreal tangents).

Of the family of curves obtained by varying k in § 2 (ii) $\frac{1}{2}(n-1)(n-2)$ have an acnode.

The two $(n-1)$-ics

$$\Sigma l_t/(l_tx+m_ty+1)=0, \quad \Sigma m_t/(l_tx+m_ty+1)=0$$

meet at these $\frac{1}{2}(n-1)(n-2)$ acnodes and at the $\frac{1}{2}n(n-1)$ intersections of the asymptotes.

The $n(n-1)$ points of contact of tangents to § 2 (ii) drawn in any given direction trace out a fixed $(n-1)$-ic through these acnodes and intersections of the asymptotes as k varies. The centroid of these points of contact coincides with the centroid of the intersections of the asymptotes, being independent of k and of the given direction.

If we take $k=\pm\epsilon$, where ϵ is a very small positive constant, the n-ic of § 2 (ii) is one of two curves each of which approximates very closely to the n asymptotes. The two curves have each a circuit composed of n infinitely extending branches when n is odd, and $\frac{1}{2}n$ circuits composed of two such branches apiece when n is even. The two curves have between them $\frac{1}{2}(n-1)(n-2)$ closed ovals. Lines can be found meeting each curve in n real points.

If we now suppose ϵ to increase, the two curves change their shape. The $\frac{1}{2}(n-1)(n-2)$ ovals shrink up one after another into an acnode and disappear, till finally the n infinitely extending branches alone are left. The curve can be projected into a closed curve or into one with a single asymptote according as n is even or odd.

If p of the asymptotes are concurrent at O, some of the above remarks require modification. For instance, the number of acnodal curves of the family is reduced by $\frac{1}{2}(p-1)(p-2)$. Taking the polar equation of the family with O as pole, and differentiating the equation partially with respect to the radius vector, we get an $(n-p)$-ic. Hence we have the result:—

"An n-ic has n n-point tangents, their points of contact lying on a given line. Of these n tangents $n-p$ are given, while the rest pass through a given point O. Then the points of contact of tangents from O lie on a fixed $(n-p)$-ic".

§ 4. Suppose now that an n-ic has not only n real n-point tangents, whose points of contact are collinear, but has also a pair of conjugate unreal n-point tangents. After a suitable projection we may suppose the chord of contact of the real n-point tangents to be at infinity, the two unreal n-point tangents to be $y=\pm i(x-2c)$, and their chord of contact to be $x=c$. First take n *odd*. When we put $i(x-2c)$ for y in § 2 (ii), the left-hand side must become $k-p(x-c)^n$. This expression can be factorized, and, identifying the factors with $l_1x+m_1i(x-2c)+1$, etc., we see that the equation of the curve must become the result of dividing by $b-c$ the equation

$$\Pi\{(c-b\cos 2\pi s/n)x+(b\sin 2\pi s/n)y+b^2-c^2\}=b^n(b^n-c^n)\dots(\mathrm{i}),$$

where $s=1, 2, \dots, n$. This may be put in the form

$$\begin{aligned}&2^n b^n(b^n-c^n)+b^n(x+iy)^n+b^n(x-iy)^n\\ &-\{cx+b^2-c^2+[(b^2-c^2)(b^2-c^2+2cx-x^2)-b^2y^2]^{\frac{1}{2}}\}^n\\ &-\{cx+b^2-c^2-[(b^2-c^2)(b^2-c^2+2cx-x^2)-b^2y^2]^{\frac{1}{2}}\}^n=0\dots(\mathrm{ii}).\end{aligned}$$

The lines through the points

$$(c+b\cos 2\pi s/n,\ b\sin 2\pi s/n)\dots\dots\dots\dots(\mathrm{iii}),$$

perpendicular to the lines joining them to $(2c, 0)$, are real n-point tangents of the curve. One of them (given by $s=n$) is $x=b+c$. The points (iii) are the vertices of a regular polygon inscribed in the circle

$$(x-c)^2+y^2=b^2,$$

and the real n-point tangents of the n-ic all touch a conic with foci $(2c, 0)$, $(0, 0)$ of which this circle is auxiliary circle. The lines $y=\pm ix$ pass through the points of contact of the two unreal n-point tangents.

The shape of the curve (i) depends on the ratio $b:c$. It degenerates if $(0, 2c)$ lies on a line joining two vertices of the regular polygon.

If $c=0$, the curve has an acnode at the origin. The real n-point tangents touch $x^2+y^2=b^2$ at the vertices of a regular polygon. The curve has the symmetry of a regular n-sided polygon; as is also evident from its polar equation

$$2^{n-1}b^n + r^n \cos n\theta = b^n + {}^nC_2 b^{n-2}(b^2-r^2) + {}^nC_4 b^{n-4}(b^2-r^2)^2 + \dots .$$

If $c=b$, the curve is

$$(2b-x)\left({}^nC_1 x^{n-1} - {}^nC_3 x^{n-3}y^2 + {}^nC_5 x^{n-5}y^4 - \dots\right) = 2^{n-1}b^n,$$

or in polar coordinates

$$r^{n-1}(2b - r\cos\theta)\sin n\theta = 2^{n-1}b^n \sin\theta.$$

If $c=-b$, the curve becomes

$$r^n \cos n\theta + 2^{n-1}b^n = 0.$$

The curve has the symmetry of a regular n-sided polygon. The n real n-point tangents are all concurrent.

Now take n *even*. We find that the curve is

$$\Pi\left\{[c - b\cos(2s+1)\pi/n]\,x + [b\sin(2s+1)\pi/n]\,y + b^2 - c^2\right\} = b^n(b^n+c^n) \quad\text{.....................(iv)},$$

where $s=1, 2, \dots, n$. This may be put in the form

$$\begin{aligned}2^n b^n(b^n+c^n) &= b^n(x+iy)^n + b^n(x-iy)^n \\ &+ \left\{cx + b^2 - c^2 + [(b^2-c^2)(b^2-c^2+2cx-x^2) - b^2y^2]^{\frac{1}{2}}\right\}^n \\ &+ \left\{cx + b^2 - c^2 - [(b^2-c^2)(b^2-c^2+2cx-x^2) - b^2y^2]^{\frac{1}{2}}\right\}^n \quad\text{...(v)}.\end{aligned}$$

As before, the real n-point tangents are the lines through the vertices

$$[c + b\cos(2s+1)\pi/n,\ b\sin(2s+1)\pi/n]$$

of a regular polygon perpendicular to the lines joining these vertices to $(2c, 0)$. If $c=0$, the curve becomes

$$2^{n-1}b^n - r^n\cos n\theta = b^n + {}^nC_2 b^{n-2}(b^2-r^2) + {}^nC_4 b^{n-4}(b^2-r^2)^2 + \dots,$$

and has an acnode at the pole. If $c=\pm b$, the curve is

$$r^n \cos n\theta = 2^{n-1}b^n.$$

So far we have assumed that the unreal n-point tangents do not intersect on the chord of contact of the n real n-point tangents. If they do, we show in a similar manner that the n-ic can be projected into

$$\Pi\left\{x - \sin 2\pi s/n,\ y - a\cos 2\pi s/n\right\} + a^n = 0,$$

i.e.,

$$2^n a^n = (a+iy)^n + (a-iy)^n - [x + (x^2-y^2-a^2)^{\frac{1}{2}}]^n - [x - (x^2-y^2-a^2)^{\frac{1}{2}}]^n,$$

when n is odd; and into

$$\Pi\{x-\sin(2s+1)\pi/n,\ y-a\cos(2s+1)\pi/n\}=a^n,$$

i.e.,

$$2^n a^n=(a+iy)^n+[x+(x^2-y^2-a^2)^{\frac{1}{2}}]^n+[x-(x^2-y^2-a^2)^{\frac{1}{2}}]^n,$$

when n is even. These curves have $y=\pm ia$ as unreal n-point tangents.

§ 5. Suppose now that n is even, and that an n-ic has three real n-point tangents forming a triangle ABC and touching at D, E, F, which are not collinear. Then a conic touches BC, CA, AB at D, E, F (§ 1)*. It readily follows that every real n-point tangent must touch this conic at its point of contact with the n-ic.

Suppose that there are n such real n-point tangents, which may be taken as the lines of § 2 (i), and that the conic which they touch at their points of contact with the n-ic is $u=0$. Then the equation of the n-ic is

$$(l_1x+m_1y+1)(l_2x+m_2y+1)\dots(l_nx+m_ny+1)=ku^{\frac{1}{2}n}\dots\text{(i)}.$$

For assume this is the case. Then there cannot be another real n-point tangent, as is seen by projecting it (supposing it exists) so that it becomes $y=0$, touching at infinity, while u becomes $1-xy$.

Now suppose we had an $(n+2)$-ic with the $n+2$ real $(n+2)$-point tangents $a_1=0$, $a_2=0$, ..., $a_{n+2}=0$, touching the conic $u=0$ at their points of contact with the $(n+2)$-ic. Using only the fact that they have two-point contact with the $(n+2)$-ic, we readily prove that the equation of the $(n+2)$-ic takes the form

$$a_1a_2\dots a_{n+2}=uv,$$

where $v=0$ is some n-ic. Then, since $a_1=0$, $a_2=0$, ..., $a_{n+2}=0$ have 2-point contact with $u=0$, they have n-point contact with $v=0$. But we have just proved that this is impossible unless $v=ku^{\frac{1}{2}n}$. Now use induction.

The curve (i) has no real inflexion other than the points of contact of the n real n-point tangents. It can have no multiple point other than an acnode. This may be seen by projecting $u=0$ into $x^2\pm y^2=1$ and the multiple point or inflexion into the origin.

Suppose that $u=0$ is projected into an ellipse. Then, if $k=\pm\epsilon$, where ϵ is a small positive constant, the curve (i)

* An n-ic having n-point contact at D, E and at least $(n-1)$-point contact at F has necessarily n-point contact at F.

is one of two curves each approximating closely to the lines of § 2 (i). Each curve has n circuits composed of two open branches apiece, and one has also $\frac{1}{4}n(n-2)$ ovals and the other has $\frac{1}{4}(n-2)^2$ ovals. The former curve has n ovals, each of which touches $u=0$ at one of its points of contact with the n-ic and shrinks up into this point of contact as ϵ increases. The latter curve has an oval touching $u=0$ at its n points of contact and approaching $u=0$ as ϵ increases. The remaining $\frac{1}{2}n(n-5)$ ovals of the two curves $(n>5)$ shrink up one after another into an acnode and disappear.

§ 6. If an n-ic has two real n-point tangents, we may take them as $z+y=0$ and $z+x=0$, their points of contact as $(1, 0, 0)$ and $(0, 0, 1)$, and any point as $(0, 0, 1)$. Then the equation of the n-ic becomes

$$z(x^{n-1}+x^{n-2}y+\ldots+y^{n-1})+xy(x^{n-2}+x^{n-3}y+\ldots+y^{n-2})$$
$$=(z+x)(z+y)u \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

where u is homogeneous of degree $n-2$ in x, y, z.

If $(0, 0, 1)$ is an $(n-1)$-ple point, $u\equiv 0$; as pointed out in Paper I.

If $(0, 0, 1)$ is an $(n-2)$-ple point, u does not involve z. Eliminating u between (i), and the result of differentiating (i) partially with respect to z, we see that the points of contact of the tangents to (i) from $(0, 0, 1)$ lie on the result of dividing

$$y^n(x+z)^2=x^n(y+z)^2$$

by $x-y$. This curve is the locus of the point of contact of tangents from the $(n-2)$-ple point of all n-ics with a given $(n-2)$-ple point and two given n-point tangents at given points. If n is odd, the curve is that discussed in § 5 of Paper I. If $n=2m$, the curve degenerates into

$$z(x^m\pm y^m)+xy(x^{m-1}\pm y^{m-1})=0.$$

§ 7. The curve

$$z(x^n-y^n)+xy(x^{n-1}-y^{n-1})=0$$

is the curve discussed in § 3 of Paper I. It is the curve into which can be projected any n-ic with an $(n-1)$-ple point and two real n-point tangents.* The curve

$$z(x^{n-1}+y^{n-1})+xy(x^{n-2}+y^{n-2})=0\ldots\ldots\ldots\text{(i)}$$

* If $n=2m$, while the n-ic has an $(n-1)$-ple point and two tangents, each having m-point contact at two points, the equation is

$$(1+b)^m(z+y)(x^2+ay^2)^m=(1+a)^m(z+x)(y^2+bx^2)^m;$$

which reduces to the above on putting $a=b=0$.

is the curve into which can be projected any n-ic with an $(n-1)$-ple point C, having DA and DB as $(n-1)$-point tangents at A and B, and touching at D the harmonic conjugate of DC for DA, DB. The corresponding curve in the case in which A, B are conjugate unreal points is

$$r\cos(n-2)\,\theta = a\cos(n-1)\,\theta.$$

This is the particular case of the curve

$$r\cos\frac{p-q}{q}\,\theta = a\,\cos\frac{p}{q}\,\theta\,\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

where $p>q$, obtained by putting $p=n-1$, $q=1$. If $EOSF$ is a line such that $OS=a$, $EO=SF=aq/(p-q)$ and the angles PSF, $pPOF/q$ differ by $\frac{1}{2}\pi$, P traces out the curve (ii), O being the pole. The properties of curve (ii) are very similar to those of the curve

$$r\,\sin\frac{p-q}{q}\,\theta = a\,\sin\frac{p}{q}\,\theta\,\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

given in § 5 of Paper I. In fact, the two curves are identical if q is even, as may be seen by turning one curve through an angle $\frac{1}{2}q\pi$ about the pole in this case. If q is odd, the curve (ii) is of degree $p+q$, has a p-ple point at O, and a q-ple point at S. The curve has singularities at the circular points, whose nature is the same as that of the origin in $y^q=x^p$.

The curves (ii) and (iii) share the property that all their inflexions are collinear, that the points of contact of the tangents from a circular point lie on a line through the other circular point, that their asymptotes are concurrent at $[-a/(n-2),\,0]$, and that the points of contact of the other tangents from this point lie on the line $2r\cos\theta=a$.

Other interesting properties are that the points of contact from any point on $\theta=0$ lie on a circle, and that all such circles form a co-axial family with O and S as limiting points. Also the inflexional tangents touch the conic with O and S as foci and touching the line of inflexions. Again, the tangents to either curve at its intersections with a line perpendicular to $\theta=0$ touch a curve of the third class.

The transformation $x=re^{\theta i}$, $y=re^{-\theta i}$, $z=-a$ converts curve (iii) into

$$y^p\,(x+z)^q = x^p\,(y+z)^q\,\ldots\ldots\ldots\ldots\ldots\text{(iv)};$$

and converts curve (ii) into

$$y^p(x+z)^q + x^p\,(y+z)^q = 0\,\ldots\ldots\ldots\ldots\ldots\text{(v)},$$

when q is odd.

Similar results hold for these curves. For instance, $\lambda x + \mu y + \nu z = 0$ touches (iv) at $(uv - v,\ u - 1,\ -u + v)$, where $u = t^p$, $v = t^q$, if

$$\frac{\lambda v}{-(p-q)\,uv + pu - qv} = \frac{\mu}{u\,(-qu + pv - p + q)} = \frac{\nu}{-q\,(u-1)^2} \dots \text{(vi)}.$$

The condition for an inflexion is

$$(p-q)\,(uv + u - v - 1) = 2p\,(u - v) \dots\dots \text{(vii)}.$$

Eliminating u and v between (vi) and (vii), we get

$$(p-q)^2 (\nu - \lambda)\,(\nu - \mu) = (p+q)^2 \lambda\mu$$

as the tangential equation ot a conic touched by the inflexional tangents, by the lines $x = 0$, $y = 0$, $x + z = 0$, $y + z = 0$, and by the line of inflexions $(p-q)\,(x+y) + 2pz = 0$.

§ 8. If three n-point tangents of a real n-ic are concurrent at O (being either all real, or one real and two conjugate unreal), their points of contact are collinear. This is at once seen by taking two of the n-point tangents as $ax^2 + 2hxy + by^2 = 0$, touching at infinity, and the third as $y = 0$. If there are n such n-point tangents meeting at $(0, 0, 1)$, their points of contact are collinear (on $z = 0$, say), and the equation of the curve takes the form

$$a_0 x^n + a_1 x^{n-1} y + \dots + a_{n-1} x y^{n-1} + a_n y^n + kz^n = 0.$$

If there is another n-point tangent, we may suppose that there is a real n-point tangent touching on $y = 0$ and passing through $(0, 1, 0)$, or else that there is a pair of conjugate unreal n-point tangents touching on $y = 0$ and meeting on $x = 0$. In either case we have $a_1 = a_2 = \dots = a_{n-1} = 0$, and the curve is projectable into

$$x^n + y^n = z^n \dots\dots\dots\dots \text{(i)}.$$

This curve meets $x = 0$ in n points, at which the tangents have n-point contact and pass through $(1, 0, 0)$; and similarly for $y = 0$, $z = 0$. Of these $3n$ n-point tangents three are real with collinear points of contact if n is odd, and four are real if n is even.

The curve may be projected into an oval with two perpendicular axes of symmetry if n is even, and into a three-branched circuit, having the symmetry of the equilateral triangle, if n is odd.

The n-ic (i) has no double point, and must therefore have $3n\,(n-2)$ inflexional tangents and $\frac{1}{2}n\,(n-2)\,(n-3)\,(n+3)$ bitangents. Of the inflexional tangents $n - 2$ coincide with

each n-point tangent. Of the bitangents $\frac{1}{2}(n-2)(n-3)$ coincide with each n-point tangent. The remaining

$$\tfrac{1}{2}n^2(n-2)(n-3)$$

bitangents are

$$(-1)^{l+m}\sin^{n-1}\frac{(l+m)\pi}{n-1}z$$

$$=(-1)^l e^{h\pi i/n}\sin^{n-1}\frac{l\pi}{n-1}x+(-1)^m e^{k\pi i/n}\sin^{n-1}\frac{m\pi}{n-1}y,$$

where l, m are integers greater than zero and $l+m=2, 3, \ldots, n-2$; while $h, k=1, 2, \ldots, n$. Of these $\frac{1}{2}(n-2)(n-3)$ are real with unreal points of contact if n is odd, and $2(n-2)(n-3)$ if n is even. The rest are unreal.

The bitangents are obtained by noticing that, if $z=\lambda x+\mu y$ is a bitangent, $(\lambda t+\mu)^n-t^n-1$ has a factor which is the square of an expression of the second degree in t. This expression must be a factor of $\mu(\lambda t+\mu)^{n-1}-1$ and of $\mu t^{n-1}-\lambda$. Now equate each of $(\lambda t+\mu)\mu^{1/(n-1)}$ and $t(\mu/\lambda)^{1/(n-1)}$ to two different $(n-1)^{\text{th}}$ roots of unity, and we get equations to determine λ and μ.

ON LAPLACE'S THEOREM ON SIMULTANEOUS ERRORS.

By *L. V. Meadowcroft, B.A., M.Sc.*

A VERY remarkable theorem on simultaneous errors was enunciated by Laplace on p. 8 of the first supplement of the *Théorie Analytique des Probabilités.* No demonstration is given, the matter being dismissed with the characteristic remark, "L'analyse du no. 21 du seconde Livre conduit à ce théorème général . . ." The theorem is as follows: Suppose that n quantities x, y, z, ... are to be determined from s observational equations of the type

$$a_i x+b_i y+c_i z+\ldots-q_i=\epsilon_i,$$

where the quantities $a_i, b_i, c_i, \ldots, q_i$ are known, and $\epsilon_1, \epsilon_2, \epsilon_3, \ldots, \epsilon_s$ are the unknown errors of observation. Let values be found for x, y, z, ... by the most advantageous method, and let these values be x_1, y_1, z_1, ... Put

$$x=x_1+\xi,\ y=y_1+\eta,\ z=z_1+\zeta,\ \ldots,$$

then the probability of the simultaneous existence of $\xi, \eta, \zeta, \ldots$ as errors of the quantities to be determined is proportional to $e^{-\sigma}$, where

$$\sigma = \frac{1}{4\kappa^2} \Sigma (a_i\xi + b_i\eta + c_i\zeta + \ldots)^2.$$

Laplace assumes that positive and negative errors are equally probable, and that the function of facility of error is the same at each observation. The theorem is, however, still true with a slight modification if these assumptions are not made.

The theorem was noticed by Todhunter in his *History of the Theory of Probability* (p. 610), but a demonstration was omitted for want of space. The omission was repaired in a subsequent paper,* dealing with certain developments of Laplace's method, in which considerable importance is attached to the result. The demonstration is based on an extension of Laplace's fundamental theorem (as set out by Poisson), involving the calculation of certain multiple integrals, and is valuable as extending the general theory. The object of the the present paper is to furnish a proof from considerations based on residual errors. In spite of its simplicity it would appear that this method has not previously been employed.

I.—*Note on Multiple Integrals.*

On p. 390 of the *Théorie Analytique des Probabilités* Laplace indicates a result which was subsequently expanded by Todhunter† into a general theorem on definite integrals. The theorem is as follows: Let

$u^2 = a_1 z_1^2 + a_2^2 (z_2 - b_1 z_1)^2 + a_3^2 (z_3 - b_2 z_2)^2 + \ldots + a_n^2 (z_n - b_{n-1} z_{n-1})^2$; let e^{-u^2} be integrated with respect to each of the $n-1$ variables $z_1, z_2, \ldots, z_{n-1}$, between the limits $-\infty$ and $+\infty$: then the result will be

$$\frac{\gamma \pi^{\frac{1}{2}(n-1)}}{a_1 a_2 \ldots a_{n-1} a_n} e^{-\gamma^2 z_n^2},$$

where

$$\frac{1}{\gamma^2} = \frac{1}{a_n^2} + \frac{b_{n-1}^2}{a_{n-1}^2} + \frac{b_{n-2}^2 b_{n-1}^2}{a_{n-2}^2} + \ldots + \frac{b_1^2 b_2^2 \ldots b_{n-1}^2}{a_1^2}.$$

Some extensions were given in a subsequent paper.

In the following analysis it is necessary to evaluate multiple integrals of the type

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots e^{-\Sigma j_i \{a_i(x_1+u)+b_i(y_1+v)+c_i(z_1+w)+\ldots-q_i-k_i\}^2} du\, dv\, dw \ldots,$$

* *Cambridge Philosophical Transactions*, vol. xi.
† *History of the Theory of Probability*, p 594.

where $x_1, y_1, z_1, \ldots$ are the values of $u, v, w, \ldots$ which make $\Sigma j_i(a_iu + b_iv + c_iw \ldots - q_i - k_i)^2$ a minimum, and the object of the present note is to present the integration in a symmetrical form by means of Gauss's notation. So far as is known the results have not previously been presented in this form, and certain of them are believed to be new.

For the sake of simplicity only three variables u, v, w will be taken. We may write

$$\Sigma j_i \{a_1(x_1 + u) + b_i(y_1 + v) + c_1(z_1 + w) - q_i - k_i\}^2$$
$$= \Sigma j_i(a_iu + biv + c_iw - \delta_i)^2, \text{ the } \delta\text{'s being independent of } u, v, w,$$
$$= u^2 \Sigma a_i^2 j_i + v^2 \Sigma b_i^2 j_i + v^2 \Sigma c_i^2 j_i$$
$$+ 2uv \Sigma a_i b_i j_i + 2vw \Sigma b_i c_i j_i + 2wu \Sigma c_i a_i j_i$$
$$- 2u \Sigma a_i \delta_i j_i - 2v \Sigma b_i \delta_i j_i - 2w \Sigma c_i \delta_i j_i + \Sigma \delta_i^2 j_i \ \ldots\ldots\ldots\ldots (1).$$

Suppose that this expression is equal to $P_1 + (Q_1w + R_1)^2$, where P_1 and R_1 are functions of u and v, and Q_1 is a numerical factor. Then, integrating with respect to w, we find

$$\int_{-\infty}^{\infty} e^{-P_1-(Q_1w+R_1)^2} dw = \frac{\sqrt{\pi}}{Q_1} e^{-P_1}.$$

Now $$Q_1^2 = \Sigma c_i^2 j_i,$$
$$Q_1R_1 = \Sigma c_i j_i (a_iu + b_iv - \delta_i),$$
and $$P_1 + R_1^2 = \Sigma j_i (a_iu + b_iv - \delta_i)^2.$$
Therefore

$$P_1 = \Sigma j_i (a_iu + b_iv - \delta_i)^2 - \frac{[\Sigma c_i j_i (a_iu + b_iv - \delta_i)]^2}{\Sigma c_i^2 j_i}.$$

On expanding this expression we readily find that

$$P_1 = u^2 \left[\Sigma a_i^2 j_i - \frac{(\Sigma a_i c_i j_i)^2}{\Sigma c_i^2 j_i}\right] + v^2 \left[\Sigma b_i^2 j_i - \frac{(\Sigma b_i c_i j_i)^2}{\Sigma c_i^2 j_i}\right]$$
$$+ 2uv \left[\Sigma a_i b_i j_i - \frac{\Sigma a_i c_i j_i \Sigma b_i c_i j_i}{\Sigma c_i^2 j_i}\right] - 2u \left[\Sigma a_i \delta_i j_i - \frac{\Sigma a_i c_i j_i \Sigma c_i \delta_i j_i}{\Sigma c_i^2 j_i}\right]$$
$$- 2v \left[\Sigma b_i \delta_i j_i - \frac{\Sigma b_i c_i j_i \Sigma c_i \delta_i j_i}{\Sigma c_i^2 j_i}\right] + \left[\Sigma \delta_i^2 j_i - \frac{(\Sigma c_i \delta_i j_i)^2}{\Sigma c_i^2 j_i}\right] \ldots (2).$$

On comparing this expression with (1) we see at once that it is derived from it in a symmetrical manner. The terms in w have disappeared, and the coefficients of the remaining terms are all altered in the same manner. If we write

$$[a_i a_i j_i . 1] = \Sigma a_i^2 j_i - \frac{(\Sigma a_i c_i j_i)^2}{\Sigma c_i^2 j_i},$$
$$[a_i b_i j_i . 1] = \Sigma a_i b_i j_i - \frac{\Sigma a_i c_i j_i \Sigma b_i c_i j_i}{\Sigma c_i^2 j_i},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

we have

$$P_1 = u^2[a_ia_ij_i.1] + v^2[b_ib_ij_i.1] + 2uv[a_ib_ij_i.1]$$
$$- 2u[a_i\delta_ij_i.1] - 2v[b_i\delta_ij_i.1] + [\delta_i\delta_ij_i.1] \ldots (3).$$

Hence, on integrating with respect to v,

$$\int_{-\infty}^{\infty} e^{-P_1}dv = \frac{\sqrt{\pi}}{Q_2}e^{-P_2},$$

where $$Q_2^2 = [b_ib_ij_i.1] \quad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and $$P_2 = u^2[a_ia_ij_i.2] - 2u[a_i\delta_ij_i.2] + [\delta_i\delta_ij_i.2],$$

if we write

$$[a_ia_ij_i.2] = [a_ia_ij_i.1] - \frac{[a_ib_ij_i.1]^2}{[b_ib_ij_i.1]},$$

$$[a_i\delta_ij_i.2] = [a_i\delta_ij_i.1] - \frac{[a_ib_ij_i.1][b_i\delta_ij_i.1]}{[b_ib_ij_i.1]},$$

$$[\delta_i\delta_ij_i.2] = [\delta_i\delta_ij_i.1] - \frac{[b_i\delta_ij_i.1]^2}{[b_ib_ij_i.1]}.$$

Finally, on integrating with respect to u, we have

$$\int_{-\infty}^{\infty} e^{-P_2}du = \frac{\sqrt{\pi}}{Q_3}e^{-P_3},$$

where $$Q_3^2 = [a_ia_ij_i.2] \quad \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

and $$P_3 = [\delta_i\delta_ij_i.3].$$

It is easily seen that

$$P_3 = \text{minimum value of } \Sigma j_i\{a_i(x_1+u)+b_i(y_1+v)+c_i(z_1+w)-q_i-k_i\}^2$$
$$= \Sigma j_i(a_ix_1 + b_iy_1 + c_iz_1 - q_i - k_i)^2.$$

Hence the triple integral

$$= \frac{\pi^{\frac{3}{2}}}{Q_1Q_2Q_3} e^{-[\delta_i\delta_ij_i.3]}$$

$$= \left\{\frac{\pi^3}{[c_ic_ij_i][b_ib_ij_i.1][a_ia_ij_i.2]}\right\}^{\frac{1}{2}} e^{-\Sigma j_i(a_ix_1+b_iy_1+c_iz_1-q_i-k_i)^2}$$

$$= \left[\frac{\pi^3}{\Delta}\right]^{\frac{1}{2}} e^{-\Sigma j_i(a_ix_1+b_iy_1+c_iz_1-q_i-k_i)^2},$$

where $$\Delta = \begin{vmatrix} \Sigma a_i^2 j_i & \Sigma a_ib_ij_i & \Sigma a_ic_ij_i \\ \Sigma a_ib_ij_i & \Sigma b_i^2 j_i & \Sigma b_ic_ij_i \\ \Sigma a_ic_ij_i & \Sigma b_ic_ij_i & \Sigma c_i^2 j_i \end{vmatrix}.$$

The result is readily extended by induction to the case of an n-tuple integral with variables $u, v, w, \ldots$, its value being

$$\left[\frac{\pi^n}{\Delta}\right]^{\frac{1}{2}} e^{-\Sigma j_i (a_i x_1 + b_i y_1 + c_i z_1 + \ldots - q_i - k_i)^2} \ldots\ldots\ldots\ldots (6).$$

Similarly we shall find that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots e^{-\Sigma j_i (a_i u + b_i v + c_i w \ldots)^2} du\, dv\, dw \; (n \text{ variables})$$

$$= \left[\frac{\pi^n}{\Delta}\right]^{\frac{1}{2}} \ldots\ldots (7).$$

The connection with Todhunter's theorem is obvious, and indeed the latter can be derived as a particular case. A comparison in particular cases would, no doubt, give rise to algebraical identities. Thus, for two variables

$$\frac{1}{a_3^2} + \frac{b_2^2}{a_2^2} + \frac{b_1^2 b_2^2}{a_1^2} = \frac{1}{a_1^2 a_2^2 a_3^2} \begin{vmatrix} a_1^2 + a_2^2 b_1^2, & -a_2^2 b_1 \\ -a_2^2 b_1, & a_2^2 + a_3^2 b_2^2 \end{vmatrix},$$

and for three variables

$$\frac{1}{a_4^2} + \frac{b_3^2}{a_3^2} + \frac{b_2^2 b_3^2}{a_2^2} + \frac{b_1^2 b_2^2 b_3^2}{a_1^2}$$

$$= \frac{1}{a_1^2 a_2^2 a_3^2 a_4^2} \begin{vmatrix} a_1^2 + a_2^2 b_1^2, & -a_2^2 b_1, & 0 \\ -a_2^2 b_1, & a_2^2 + a_3^2 b_2^2, & -a_3^2 b_2 \\ 0, & -a_3^2 b_2, & a_3^2 + a_4^2 b_3^2 \end{vmatrix},$$

results which can be easily be generalized. Further identities may be obtained by performing the integrations in different orders and equating the results.

It should be noted that in the course of the above analysis it has been shown that the minimum value of

$$\Sigma j_i (a_i u + b_i v + c_i w + \ldots - \delta_i)^2$$

is $[\delta_i \delta_i j_i . n]$, a known result.

Since $x_1, y_1, z_1, \ldots$ satisfy the normal equations

$$\left.\begin{array}{l} x_1 \Sigma a_i^2 j_i + y_1 \Sigma a_i b_i j_i + z_1 \Sigma a_i c_i j_i + \ldots = \Sigma (q_i + k_i)\, a_i j_i \\ x_1 \Sigma a_i b_i j_i + y_1 \Sigma b_i^2 j_i + z_1 \Sigma b_i c_i j_i + \ldots = \Sigma (q_i + k_i)\, b_i j_i \\ x_1 \Sigma a_i c_i j_i + y_1 \Sigma b_i c_i j_i + z_1 \Sigma c_i^2 j_i + \ldots = \Sigma (q_i + k_i)\, c_i j_i \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \ldots (8),$$

it is clear that $[a_i \delta_i j_i] = [b_i \delta_i j_i . 1] = \ldots = 0$, and that many of the terms in the above analysis might have been omitted.

They were retained for the sake of symmetry, and also because it is not essential that the values of x_1, y_1, z_1, ... should be introduced at all. By using determinants the same method may be employed in the case of a general quadratic expression, provided that the coefficients satisfy certain conditions.

II.—*On Laplace's Theorem.*

If $\phi_i(x)$ is the function of facility of error at the i^{th} observation, we have

$$\int_{-\infty}^{\infty} \phi_i(x)\,dx = 1,$$

$$\int_{-\infty}^{\infty} x\phi_i(x)\,dx = k_i,$$

$$\int_{-\infty}^{\infty} x^2\phi_i(x)\,dx = k_i',$$

so that k_1, k_2, ..., k_s are the mean errors, and k_1', k_2', ..., k_s' the errors of mean square. Then x_1, y_1, z_1, ..., the most advantageous values of x, y, z, are given by the normal equations (8), where

$$\frac{1}{j_i} = h_i^2 = \tfrac{1}{2}(k_i' - k_i^2).$$

Let us suppose, in the first place, that

$$\phi_i(x) = \frac{1}{2h_i\sqrt{\pi}}\, e^{-\{(x-k_i)^2/4h_i^2\}} \quad \ldots\ldots\ldots\ldots (9).$$

It is easily verified that

$$\frac{1}{2h_i\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\{(x-k_i)^2/4h_i^2\}}\,dx = 1,$$

$$\frac{1}{2h_i\sqrt{\pi}} \int_{-\infty}^{\infty} xe^{-\{(x-k_i)^2/4h_i^2\}}\,dx = k_i,$$

$$\frac{1}{2h_i\sqrt{\pi}} \int_{-\infty}^{\infty} x^2e^{-\{(x-k_i)^2/4h_i^2\}}\,dx = k_i',$$

so that this function has the same mean error and error of mean square as the more general function referred to above. If now the errors in x_1, y_1, z_1, ... are ξ, η, ζ, ..., the true values are

$$x_1 + \xi,\ y_1 + \eta,\ z_1 + \zeta,\ \ldots,$$

and the residuals are of the form

$$a_i(x_1 + \xi) + b_i(y_1 + \eta) + c_i(z_1 + \zeta) + \ldots - q_i = \epsilon_i.$$

We have therefore to determine the *à posteriori* probability of this system of residuals.

This is proportional to $e^{-\Sigma\{(\epsilon_i-k_i)^2/4h_i^2\}}$, *i.e.* to

$$e^{-\Sigma\,[\{a_i(x_1+\xi)+b_i(y_1+\eta)+c_i(z_1+\zeta)+\ldots-q_i-k_i\}^2/4h_i^2]}.$$

Let p denote the probability that the error in x_1 will lie between ξ and $\xi+d\xi$, that in y_1 between η and $\eta+d\eta$, that in z_1 between ζ and $\zeta+d\zeta$, and so on. Then, since *à priori* ξ, η, ζ, ... may have any values between $+\infty$ and $-\infty$, we have

$$p=\frac{e^{-\Sigma\frac{1}{4}j_i\{a_i(x_1+\xi)+b_i(y_1+\eta)+c_i(z_1+\zeta)+\ldots-q_i-k_i\}^2}}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots e^{-\Sigma\frac{1}{4}j_i\{a_i(x_1+u)+b_i(y_1+v)+c_i(z_1+w)+\ldots-q_i-k_i\}^2}du\,dv\,dw}$$
$$\times\, d\xi\, d\eta\, d\zeta \ldots\ldots(10).$$

Now the index of the exponential in the numerator (disregarding the negative sign)

$$=\tfrac{1}{4}\Sigma j_i(a_i\xi+b_i\eta+c_i\zeta+\ldots)^2+\tfrac{1}{4}\Sigma j_i(a_ix_1+b_iy_1+c_iz_1+\ldots-q_i-k_i)^2$$
$$+\tfrac{1}{2}\xi\Sigma j_ia_i(a_ix_1+b_iy_1+c_iz_1+\ldots-q_i-k_i)+\text{similar terms in }\eta,\zeta,\ldots$$
$$=\tfrac{1}{4}\Sigma j_i(a_i\xi+b_i\eta+c_i\zeta+\ldots)^2+\tfrac{1}{4}\Sigma j_i(a_ix_1+b_iy_1+c_iz_1+\ldots-q_i-k_i)^2,$$

since the coefficients of ξ, η, ζ, ... vanish in virtue of the equations (8).

The value of the integral in the denominator is

$$\left[\frac{2^{2n}\pi^n}{\Delta}\right]^{\frac{1}{2}}e^{-\frac{1}{4}\Sigma j_i(a_ix_1+b_iy_1+c_iz_1+\ldots-q_i-k_i)^2}\text{ by (6).}$$

Hence $$p=\frac{\sqrt{\Delta}}{2^n\pi^{\frac{1}{2}n}}e^{-\frac{1}{4}\Sigma j_i(a_i\xi+b_i\eta+c_i\zeta+\ldots)^2}d\xi\,d\eta\,d\zeta-\ldots\ \ldots(11).$$

This agrees with Todhunter's result; the numerical factor was not given by Laplace.

It remains to establish that the formula is still true as a first approximation in the general case and that it involves the same degree of approximation as is adopted throughout Laplace's investigations.

If the error laws are $\phi_1(x)$, $\phi_2(x)$, $\phi_3(x)$, ..., $\phi_s(x)$ with mean errors k_1, k_2, k_3, ..., k_s and errors of mean square k_1', k_2', k_3', ..., k_s' the first part of the proposition is obvious, since the particular error laws (9), which have been adopted, have the same mean errors and errors of mean square; they are in fact the natural first approximations which modern theory suggests. Moreover, it is not difficult to see that the degree of approximation is the same as results from the application of Laplace's fundamental theorem, and Todhunter's extension of it, to linear combinations of the observational equations. In

Laplace's analysis the results are exact until approximate values are taken for the integrals

$$\int_b^a \phi_i(z) \cos \gamma_i xz \, dz,$$

$$\int_b^a \phi_i(z) \sin \gamma_i xz \, dz.$$

The approximate values taken for these integrals are $1 - \frac{x^2 \gamma_i k_i'}{2}$ and $x\gamma_i k_i$ respectively (higher moments than the second being ignored). Now $\frac{1}{2h_i \sqrt{\pi}} e^{-\{(x-k_i)^2/4h_i^2\}}$ has the same area as $\phi_i(x)$ and the same first and second moments about the origin, so that if

$$\phi_i(x) = \frac{1}{2h_i \sqrt{\pi}} e^{-\{(x-k_i)^2/4h_i^2\}} + f_i(x),$$

we have

$$\int_{-\infty}^{\infty} f_i(x) \, dx = 0,$$

$$\int_{-\infty}^{\infty} x f_i(x) \, dx = 0,$$

$$\int_{-\infty}^{\infty} x^2 f_i(x) \, dx = 0,$$

and the functions $f_i(x)$ do not enter in any way into Laplace's fundamental result.

It is a matter of some interest to show that the formula for p leads to Laplace's result for an error of given amount in x_1 when the errors in y_1, z_1, ... are unrestricted and may have any values between $+\infty$ and $-\infty$. Thus with three variables the probability that the error in x_1 will lie between ξ and $\xi + d\xi$

$$= d\xi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p \, d\eta \, d\zeta$$

$$= d\xi \, \frac{Q_1 Q_2 Q_3}{\pi^{\frac{3}{2}}} \cdot \frac{\sqrt{\pi}}{Q_1} \cdot \frac{\sqrt{\pi}}{Q_2} e^{-Q_3^2 \xi^2}$$

$$= \quad \frac{Q_3}{\sqrt{\pi}} e^{-Q_3^2 \xi^2} d\xi.$$

This result easily follows by observing that the course of the integration is the same as for the integration in the denominator of p in (10), the final index of the exponential being the same as the term in u^2 in $P_3 + (Q_3 u + R_3)^2$.

Now

$$Q_3^2 = \frac{\Delta}{2^6 Q_1^2 Q_2^2} = \frac{\Delta}{2^2 [\Sigma b_i^2 j_i \Sigma c_i^2 j_i - (\Sigma b_i c_i j_i)^2]}.$$

Laplace's result is $\frac{1}{2\kappa\sqrt{\pi}} e^{-\xi^2/4\kappa^2} d\xi$, where $\kappa^2 = \lambda$ and λ is given by

$$1 = \lambda \Sigma a_i^2 j_i + \mu \Sigma a_i b_i j_i + \nu \Sigma a_i c_i j_i,$$
$$0 = \lambda \Sigma a_i b_i j_i + \mu \Sigma b_i^2 j_i + \nu \Sigma b_i c_i j_i,$$
$$0 = \lambda \Sigma a_i c_i j_i + \mu \Sigma b_i c_i j_i + \nu \Sigma c_i^2 j_i.$$

The results are readily seen to be identical.

Similarly if it is desired to determine the probability that the errors in any assigned number of the variables shall have given values, the errors in the remainder being unrestricted, the result can at once be written down in a symmetrical form by using the formulæ derived in § 1.

It is of course clear from the modern theory of errors that the method adopted above of working with the particular exponential laws (9) by the method of residuals will enable us to derive not only this particular theorem of Laplace's, but also the whole of his results. In most text-books on the subject, however, only the normal error law is employed in this connection. It can hardly be doubted that many advantages would accrue if this extension were more generally adopted; in particular it leads to a direct proof of the method of least squares, whereas in Laplace's method it is necessary to be content with an *à posteriori* verification.

On pages 19—21 of the first supplement Laplace observes that all his analysis rests on the assumption that positive and negative errors are equally probable, and proposes to show that this limitation does not practically affect the value of his results. Todhunter* expresses the opinion that no great conviction would be gained from the investigation. It is, however, not known how Laplace demonstrated his theorem on simultaneous errors. It can hardly be doubted that if it had involved the same type of analysis as Todhunter's proof he would have given some indication of his method. On the other hand portions of the first supplement itself suggest that Laplace may have used the method of residuals, whilst on page 390 of the *Théorie Analytique* . . . a result on integration is given which has been utilised in reducing the multiple integral in (10). It may well be, therefore, that Laplace had a more solid foundation for his generalisation than his analysis would suggest.

* *History of the Theory of Probability*, p. 611.

FOUR-VECTOR ALGEBRA AND ANALYSIS.

Part II.*

By *C. E. Weatherburn, M.A., D.Sc.*

Four-Vector Analysis.

A.—*The differential Operations.*

§ 15. *Differentiation of vectors. Point functions.* Let $\mathfrak{a}$ be a four-vector function of the scalar variables s, t, ..., and $\delta\mathfrak{a}$ the increment in the vector due to an increment δs in the first variable, the values of the other variables remaining unaltered. Then the limiting value of the quotient $\delta\mathfrak{a}/\delta s$ as δs tends to zero is called the *partial derivative* of $\mathfrak{a}$ with respect to s, and is written

$$\frac{\partial \mathfrak{a}}{\partial s} = \underset{\delta s \to 0}{\text{Lt}} \frac{\delta \mathfrak{a}}{\delta s}.$$

This is itself a four-vector, and in general also a function of the same variables. Its derivatives with respect to s, t, ... are written

$$\frac{\partial^2 \mathfrak{a}}{\partial s^2}, \frac{\partial^2 \mathfrak{a}}{\partial t\, \partial s}, \ldots,$$

as in the case of algebraic functions. And what has just been written in connection with four-vector functions is equally true of six-vectors.

Products of vectors are differentiated according to the same rules as in the algebraic calculus. Thus

$$\frac{\partial}{\partial s}(\mathfrak{a} \cdot \mathfrak{b}) = \frac{\partial \mathfrak{a}}{\partial s} \cdot \mathfrak{b} + \mathfrak{a} \cdot \frac{\partial \mathfrak{b}}{\partial s},$$

$$\frac{\partial}{\partial s}(\mathfrak{a} \times \mathfrak{b}) = \frac{\partial \mathfrak{a}}{\partial s} \times \mathfrak{b} + \mathfrak{a} \times \frac{\partial \mathfrak{b}}{\partial s},$$

$$\frac{\partial}{\partial s}[\mathfrak{a}\mathfrak{F}\mathfrak{b}] = \left[\frac{\partial \mathfrak{a}}{\partial s} \mathfrak{F}\mathfrak{b}\right] + \left[\mathfrak{a} \frac{\partial \mathfrak{F}}{\partial s} \mathfrak{b}\right] + \left[\mathfrak{a}\mathfrak{F} \frac{\partial \mathfrak{b}}{\partial s}\right],$$

and so on. In the case of cross products the order of the factors must not be changed without altering the sign of that term. The above formulæ are simple consequences of the distributive law.

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* Part I. was published in vol. xlix., pp. 155–176.

When the function considered (whether a scalar or vector) is a function of the coordinates $x, y, z, l=ict$ of a point in Minkowski's four-dimensional space, and has a definite and unique value at each point, it may be called a single-valued *point-function.* Such a function ϕ will possess derivatives

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \ldots,$$

with respect to the coordinates of the point.

When by the orthogonal transformation of § 1 the variables x, y, z, l are transformed to x', y', z', l' the function ϕ becomes a function of the latter variables. To find how its derivatives with respect to these are related to those with respect to the former set we notice that

$$\begin{aligned}\frac{\partial\phi}{\partial x'} &= \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial\phi}{\partial y}\frac{\partial y}{\partial x'} + \frac{\partial\phi}{\partial z}\frac{\partial z}{\partial x'} + \frac{\partial\phi}{\partial l}\frac{\partial l}{\partial x'} \\ &= \frac{\partial\phi}{\partial x}a_{11} + \frac{\partial\phi}{\partial y}a_{12} + \frac{\partial\phi}{\partial z}a_{13} + \frac{\partial\phi}{\partial l}a_{14},\end{aligned}$$

while similar values may be written down for $\frac{\partial\phi}{\partial y'}$, $\frac{\partial\phi}{\partial z'}$, and $\frac{\partial\phi}{\partial l'}$. If then we form the matrix of operators

$$d = \left\| \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial l} \right\|,$$

the above formulæ show that

$$\begin{aligned} d' &\equiv \left\| \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}, \frac{\partial}{\partial l'} \right\| \\ &= \left\| \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial l} \right\| \left\| \begin{matrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & . & . & . \\ a_{13} & . & . & . \\ a_{14} & . & . & . \end{matrix} \right\| = d\bar{A} \ldots\ldots (1).\end{aligned}$$

That is to say the set of four operators $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial l}\right)$ is transformed like a four-vector. We shall adopt the notation†

† The symbol $\square$ was introduced by Cauchy to denote the operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \ldots + \frac{\partial^2}{\partial l^2},$$

for which we use $\square^2$. This alteration makes the notation run parallel with that for three dimensions.

$$\square = \mathfrak{i}\frac{\partial}{\partial x} + \mathfrak{j}\frac{\partial}{\partial y} + \mathfrak{k}\frac{\partial}{\partial z} + \mathfrak{h}\frac{\partial}{\partial l} \quad \text{..............(2)}$$

for this four-vector operator, which is analogous to the operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

of three dimensions. In terms of the dyadic Λ of § 9, we may write

$$\square' = \Lambda \cdot \square \quad \text{........................(1')}$$

as the equivalent of (1).

§ 16. *Four-vector gradient of a scalar point function.* Let ϕ be a scalar point function. Then by the preceding section

$$\mathfrak{i}\frac{\partial\phi}{\partial x'} + \mathfrak{j}\frac{\partial\phi}{\partial y'} + \ldots + \mathfrak{h}\frac{\partial\phi}{\partial l'} = \Lambda \cdot \left(\mathfrak{i}\frac{\partial\phi}{\partial x} + \mathfrak{j}\frac{\partial\phi}{\partial y} + \ldots + \mathfrak{h}\frac{\partial\phi}{\partial l}\right)$$

which we write briefly

$$\square'\phi = \Lambda \cdot \square\phi.$$

Thus
$$\square\phi \equiv \mathfrak{i}\frac{\partial\phi}{\partial x} + \mathfrak{j}\frac{\partial\phi}{\partial y} + \mathfrak{k}\frac{\partial\phi}{\partial z} + \mathfrak{h}\frac{\partial\phi}{\partial l} \quad \text{...........(3)}$$

is a four-vector, and by analogy with the three-dimensional function is called the *four-vector gradient* of ϕ. Its scalar product with the unit vector $\hat{\mathfrak{a}} = (a_1, a_2, a_3, a_4)$ is

$$\hat{\mathfrak{a}} \cdot \square\phi = a_1\frac{\partial\phi}{\partial x} + a_2\frac{\partial\phi}{\partial y} + a_3\frac{\partial\phi}{\partial z} + a_4\frac{\partial\phi}{\partial l} \quad \text{.........(4),}$$

and may be called the directional derivative of ϕ for the direction of $\hat{\mathfrak{a}}$. In particular

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}, \frac{\partial\phi}{\partial l}$$

are the directional derivatives along the coordinate axes.

An important case is that for which $\phi = R^n$, where $R = \sqrt{(x^2 + y^2 + z^2 + l^2)}$ is the "distance" of the point from the origin. Then

$$\frac{\partial\phi}{\partial x} = nR^{n-1}\frac{\partial R}{\partial x} = nR^{n-1}\frac{x}{R},$$

and therefore

$$\square R^n = nR^{n-2}\mathfrak{r} = nR^{n-1}\hat{\mathfrak{r}} \quad \text{..................(5),}$$

where $$\mathfrak{r} = x\mathfrak{i} + y\mathfrak{j} + z\mathfrak{k} + l\mathfrak{h}$$

is the "position four-vector" of the given point, and $\hat{\mathfrak{r}}$ the parallel unit vector. More generally, if V is any scalar function of R, we find in the same way that

$$\square V = \frac{\partial V}{\partial R}\hat{\mathfrak{r}} \quad \text{......................(6).}$$

Further if $\mathfrak{r}$ is the position four-vector of a point P, and $\mathfrak{r} + \delta\mathfrak{r}$ that of a neighbouring point P', the difference

$$\delta\mathfrak{r} = \delta x\,\mathfrak{i} + \delta y\,\mathfrak{j} + \delta z\,\mathfrak{k} + \delta l\,\mathfrak{h}$$

is also a four-vector; or, in the notation of differentials,

$$d\mathfrak{r} = dx\,\mathfrak{i} + dy\,\mathfrak{j} + dz\,\mathfrak{k} + dl\,\mathfrak{h}$$

is a four-vector. Hence for a point function ϕ

$$\begin{aligned} d\mathfrak{r}\cdot\square\phi &= (dx\,\mathfrak{i} + dy\,\mathfrak{j} + \ldots)\cdot\left(\frac{\partial\phi}{\partial x}\mathfrak{i} + \frac{\partial\phi}{\partial y}\mathfrak{j} + \ldots\right) \\ &= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \ldots + \frac{\partial\phi}{\partial l}dl \\ &= d\phi \quad \text{..(7),} \end{aligned}$$

where $d\phi$ is the increment of ϕ in passing from P to P'. This formula is analogous to

$$d\mathbf{r}\cdot\nabla\phi = d\phi$$

in the case of three dimensions.

§ 17. *Divergence and curl of a four-vector.* Similarly we may define functions analogous to the divergence and curl of an ordinary vector. If $\mathfrak{a}$ is a four-vector point function, its scalar product with the symbolic vector $\square$ is invariant with respect to an orthogonal transformation, and has the value

$$\begin{aligned} \square\cdot\mathfrak{a} &= \Sigma\,\mathfrak{i}\cdot\frac{\partial\mathfrak{a}}{\partial x} \\ &= \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z} + \frac{\partial a_4}{\partial l} \quad \text{...............(8).} \end{aligned}$$

This is sometimes called the *divergence* of the four-vector $\mathfrak{a}$, and is abbreviated div $\mathfrak{a}$. With the notation $\mathfrak{a} = (\mathbf{a}, a_4)$ we may write

$$\operatorname{div}\mathfrak{a} = \square\cdot\mathfrak{a} = \nabla\cdot\mathbf{a} + \frac{\partial a_4}{\partial l} \quad \text{..............(9).}$$

In the same way the cross product $\square \times \mathfrak{a}$ is a six-vector, which may be called the curl of $\mathfrak{a}$. Thus

$$\text{curl } \mathfrak{a} = \square \times \mathfrak{a} = \Sigma \mathfrak{i} x \times \frac{\partial \mathfrak{a}}{\partial x}$$

$$= \left(\frac{\partial a_3}{\partial y} - \frac{\partial a_2}{\partial z}, \frac{\partial a_1}{\partial z} - \frac{\partial a_3}{\partial x}, \frac{\partial a_2}{\partial x} - \frac{\partial a_1}{\partial y},\right.$$
$$\left.\frac{\partial a_4}{\partial x} - \frac{\partial a_1}{\partial l}, \frac{\partial a_4}{\partial y} - \frac{\partial a_2}{\partial l}, \frac{\partial a_4}{\partial z} - \frac{\partial a_3}{\partial l}\right).$$

With the notation $\mathfrak{a} = (\mathbf{a}, a_4)$ we may write this more briefly

$$\square \times \mathfrak{a} = \left(\nabla \times \mathbf{a},\ \nabla a_4 - \frac{\partial \mathbf{a}}{\partial l}\right) \qquad \text{.............(10).}$$

If ever it is convenient to use the expression $\mathfrak{a} \times \square$ it should be understood as $-\square \times \mathfrak{a}$. In the case of the position four-vector $\mathfrak{r}$, it is worth noticing that

$$\square \cdot \mathfrak{r} = 4, \quad \square \times \mathfrak{r} = 0 \qquad \text{.................(11).}$$

Finally we may form the cross product of $\square$ and a six-vector point function $\mathfrak{F}$, which is a four-vector. Minkowski denoted this product by lor $\mathfrak{F}$ in honour of Lorentz. Thus if Φ is the anti-selfconjugate dyadic corresponding to $\mathfrak{F}$,

$$\text{lor } \mathfrak{F} = \square \times \mathfrak{F} = -\square \cdot \Phi$$

$$= \left(\frac{\partial}{\partial y} f_{12} + \frac{\partial}{\partial z} f_{13} + \frac{\partial}{\partial l} f_{14}\right)\mathfrak{i} + \left(\frac{\partial}{\partial x} f_{21} + \frac{\partial}{\partial y} f_{23} + \frac{\partial}{\partial l} f_{24}\right)\mathfrak{j}$$
$$+ \left(\frac{\partial}{\partial x} f_{31} + \frac{\partial}{\partial y} f_{32} + \frac{\partial}{\partial l} f_{34}\right)\mathfrak{k} + \left(\frac{\partial}{\partial x} f_{41} + \frac{\partial}{\partial y} f_{42} + \frac{\partial}{\partial z} f_{43}\right)\mathfrak{h} \quad \text{...(12).}$$

With the notation $\mathfrak{F} = (\mathbf{f}_1, \mathbf{f}_2)$ this may be more conveniently expressed

$$\square \times \mathfrak{F} = \left(\nabla \times \mathbf{f}_1 + \frac{\partial \mathbf{f}_2}{\partial l},\ -\nabla \cdot \mathbf{f}_2\right) \qquad \text{........(12').}$$

§ 18. *Second order differential functions.* By operating on the above functions with the vector operator $\square$ we may form certain second order differential functions. Thus taking the divergence of $\square\phi$ we find

$$\square \cdot \square\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \phi}{\partial l^2} \qquad \text{.........(13).}$$

This is often called the *D'Alembertian* of ϕ, and is analogous to the Laplacian $\nabla \cdot \nabla \phi \equiv \nabla^2 \phi$. In agreement with this

notation, and also with the notation for the square of a vector, we shall write the expression (13) as $\Box^2\phi$. Thus

$$\Box^2\phi = \text{div grad}\,\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \ldots + \frac{\partial^2\phi}{\partial l^2} \quad \ldots\ldots(13').$$

If in the same way we form the curl of $\Box\phi$ we find that it vanishes identically. Thus

$$\Box \times \Box\phi \equiv 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

as we should expect from the symbolic product $\Box \times \Box$ appearing in the expression. In the same way it will be found that div lor $\mathfrak{F}$ vanishes identically. Thus

$$\Box \cdot \Box \times \mathfrak{F} \equiv 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(15),$$

the scalar triple product having the repeated symbolic vector $\Box$. Finally, if we take the lor of the dual vector to curl $\mathfrak{a}$, we find that it also vanishes. Thus

$$\Box \times (\Box \times \mathfrak{a})^* \equiv 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(16).$$

This agrees with § 12 Cor., according to which such a product with a repeated factor should vanish. However, lor curl $\mathfrak{a}$ does not vanish, but will be found to have the value

$$\Box \times (\Box \times \mathfrak{a}) = \Box\Box \cdot \mathfrak{a} - \Box^2\mathfrak{a} \ldots\ldots\ldots\ldots\ldots(17),$$

which agrees with (17) of § 12, and is analogous to the formula

$$\nabla \times \nabla \times \mathbf{a} = \nabla\nabla \cdot \mathbf{a} - \nabla^2\mathbf{a}$$

of three dimensions, and may be proved in the same way†.

There is one other formula which may be mentioned here. If $\mathfrak{F}$ is any six-vector

$$\Box \times \Box \times \mathfrak{F} \times (\Box \times \Box \times \mathfrak{F}^*)^* = -\Box^2\mathfrak{F} \ldots\ldots(18).$$

To prove this write $\mathfrak{F} = (\mathbf{f}_1, \mathbf{f}_2)$. Then

$$\Box \times \mathfrak{F} = \left(\nabla \times \mathbf{f}_1 + \frac{\partial \mathbf{f}_2}{\partial l},\ -\nabla \cdot \mathbf{f}_2\right),$$

and therefore

$$\Box \times \Box \times \mathfrak{F} = \left\{\nabla \times \left(\nabla \times \mathbf{f}_1 + \frac{\partial \mathbf{f}_2}{\partial l}\right),\right.$$

$$\left. -\nabla\nabla \cdot \mathbf{f}_2 - \frac{\partial}{\partial l}\left(\nabla \times \mathbf{f}_1 + \frac{\partial \mathbf{f}_2}{\partial l}\right)\right\}.$$

† Cf. the author's *Advanced Vector Analysis*, Art. 9.

Similarly

$(\square \times \square \times \mathfrak{F}^*)^*$

$$= \left\{ -\nabla\nabla\cdot\mathbf{f}_1 - \frac{\partial}{\partial l}\left(\nabla\times\mathbf{f}_2 + \frac{\partial \mathbf{f}_1}{\partial l}\right),\ \ \nabla\times\left(\nabla\times\mathbf{f}_2 + \frac{\partial \mathbf{f}_1}{\partial l}\right)\right\},$$

and the sum of these, got by adding their like components, is equal to

$$-\left(\nabla^2\mathbf{f}_1 + \frac{\partial^2 \mathbf{f}_1}{\partial l^2},\ \ \nabla^2\mathbf{f}_2 + \frac{\partial^2 \mathbf{f}_2}{\partial l^2}\right) = -\square^2(\mathbf{f}_1,\ \mathbf{f}_2) = -\square^2\mathfrak{F}.$$

It will be noticed that this formula corresponds to (26) of § 14, with each of the four-vectors replaced by the symbolic four-vector $\square$.

§ 19. *Formulæ of expansion.* It will be found convenient to have formulæ of expansion for the curl and divergence of a vector point function of the form $u\mathfrak{a}$ or $\mathfrak{a}\times\mathfrak{F}$, where u, $\mathfrak{a}$, $\mathfrak{F}$ are themselves point functions; and also for lor$(\mathfrak{a}\times\mathfrak{b})$ and the gradient of a product of functions. The formulæ at which we arrive may be expressed

$$\square\,(uv) = u\square v + v\square u \quad\ldots\ldots(19),$$

$$\square\cdot(u\mathfrak{a}) = \square u\cdot\mathfrak{a} + u\square\cdot\mathfrak{a} \quad\ldots\ldots(20),$$

$$\square\times(u\mathfrak{a}) = \square u\times\mathfrak{a} + u\square\times\mathfrak{a} \quad\ldots\ldots(21),$$

$$\square\cdot(\mathfrak{a}\times\mathfrak{F}) = \mathfrak{F}\cdot\square\times\mathfrak{a} - \mathfrak{a}\cdot\square\times\mathfrak{F} \quad\ldots\ldots(22),$$

$$\square\times(\mathfrak{a}\times\mathfrak{b}) = \mathfrak{b}\cdot\square\mathfrak{a} - \mathfrak{a}\cdot\square\mathfrak{b} + \mathfrak{a}\square\cdot\mathfrak{b} - \mathfrak{b}\square\cdot\mathfrak{a} \quad\ldots\ldots(23),$$

$$\square\,(\mathfrak{a}\cdot\mathfrak{b}) = \mathfrak{b}\cdot\square\mathfrak{a} + \mathfrak{a}\cdot\square\mathfrak{b} + \mathfrak{b}\times\square\times\mathfrak{a} + \mathfrak{a}\times\square\times\mathfrak{b} \ldots(24).$$

The first of these is obvious from the rule for differentiating a product. To prove the second we may proceed

$$\begin{aligned}\square\cdot(u\mathfrak{a}) &= \Sigma\,\mathfrak{i}\cdot\frac{\partial}{\partial x}(u\mathfrak{a})\\ &= \left(\Sigma\,\mathfrak{i}\,\frac{\partial u}{\partial x}\right)\cdot\mathfrak{a} + u\,\Sigma\,\mathfrak{i}\cdot\frac{\partial\mathfrak{a}}{\partial x}\\ &= \square u\cdot\mathfrak{a} + u\square\cdot\mathfrak{a}\end{aligned}$$

as required; and (21) may be proved in the same manner. Similarly in the case of (22) we have

$$\begin{aligned}\square\cdot(\mathfrak{a}\times\mathfrak{F}) &= \Sigma\,\mathfrak{i}\cdot\left(\frac{\partial\mathfrak{a}}{\partial x}\times\mathfrak{F} + \mathfrak{a}\times\frac{\partial\mathfrak{F}}{\partial x}\right)\\ &= \left(\Sigma\,\mathfrak{i}\times\frac{\partial\mathfrak{a}}{\partial x}\right)\cdot\mathfrak{F} - \mathfrak{a}\cdot\Sigma\,\mathfrak{i}\times\frac{\partial\mathfrak{F}}{\partial x}\\ &= \mathfrak{F}\cdot\square\times\mathfrak{a} - \mathfrak{a}\cdot\square\times\mathfrak{F},\end{aligned}$$

as was to be proved.

To establish (23) we have similarly

$$\square \times (\mathfrak{a} \times \mathfrak{b}) = \Sigma \mathfrak{i} \times \left(\frac{\partial \mathfrak{a}}{\partial x} \times \mathfrak{b} + \mathfrak{a} \times \frac{\partial \mathfrak{b}}{\partial x}\right)$$

$$= \Sigma \left(\mathfrak{i} \cdot \mathfrak{b} \frac{\partial \mathfrak{a}}{\partial x} - \mathfrak{i} \cdot \frac{\partial \mathfrak{a}}{\partial x} \mathfrak{b}\right) + \Sigma \left(\mathfrak{i} \cdot \frac{\partial \mathfrak{b}}{\partial x} \mathfrak{a} - \mathfrak{i} \cdot \mathfrak{a} \frac{\partial \mathfrak{b}}{\partial x}\right)$$

$$= \mathfrak{b} \cdot \square\mathfrak{a} - \mathfrak{a} \cdot \square\mathfrak{b} + \mathfrak{a}\square \cdot \mathfrak{b} - \mathfrak{b}\square \cdot \mathfrak{a}.$$

In this formula $\mathfrak{b} \cdot \square\mathfrak{a}$ is the scalar product of $\mathfrak{b}$ and the dyadic

$$\square\mathfrak{a} = \Sigma \mathfrak{i} \frac{\partial \mathfrak{a}}{\partial x},$$

which will be further considered in § 21. The expression may also be interpreted as

$$(\mathfrak{b} \cdot \square)\, \mathfrak{a} = \left(b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} + \ldots + b_4 \frac{\partial}{\partial l}\right) \mathfrak{a},$$

which is $\sqrt{\mathfrak{b}^2}$ times the derivative of $\mathfrak{a}$ in the direction of $\mathfrak{b}$. Formula (24) may be proved in like manner.

In place of (23) we could write, by §§ 10—11,

$$\square \times (\mathfrak{a} \times \mathfrak{b}) = - \square \cdot (\mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a})$$

$$= \square \cdot (\mathfrak{b}\mathfrak{a} - \mathfrak{a}\mathfrak{b}) \quad \ldots\ldots\ldots\ldots(25),$$

which will be further considered in the next part of this paper, where also another formula will be found which is the equivalent of (24).

§ 20. *The function R^n.* We saw in § 16 that if R is the "distance" of a variable point from the origin

$$\square R^n = nR^{n-2}\mathfrak{r},$$

where $\mathfrak{r}$ is the position four-vector of the variable point, and $R^2 = \mathfrak{r}^2$. Hence by (20)

$$\square^2 R^n = \square \cdot \square R^n = n\square \cdot (R^{n-2}\mathfrak{r})$$

$$= n\,(\square R^{n-2} \cdot \mathfrak{r} + R^{n-2}\square \cdot \mathfrak{r}).$$

Then since $\square \cdot \mathfrak{r} = 4$ the equation reduces to

$$\square^2 R^n = n\,\{(n-2)\, R^{n-4}\mathfrak{r}^2 + 4R^{n-2}\}$$

$$= n\,(n+2)\, R^{n-2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(26).$$

If $n = 0$ or -2 this expression vanishes. The first value for n makes R^n a constant. Taking the second value we have

$$\square^2 \frac{1}{R^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(27),$$

showing that $1/R^2$ is a solution of D'Alembert's equation.

More generally, if V is a scalar function of R, we have by (6)

$$
\begin{aligned}
\square^2 V &= \square \cdot \left(\frac{\partial V}{\partial R}\hat{\mathfrak{r}}\right) = \square \cdot \left(\frac{V'}{R}\mathfrak{r}\right) \\
&= \square\left(\frac{V'}{R}\right) \cdot \mathfrak{r} + 4\frac{V'}{R} \\
&= \left(\frac{V''}{R} - \frac{V'}{R^2}\right)\hat{\mathfrak{r}} \cdot \mathfrak{r} + 4\frac{V'}{R} \\
&= V'' + \frac{3V'}{R} \qquad\qquad (28),
\end{aligned}
$$

dashes denoting differentiations with respect to R.

B.—*Dyadics involving* $\square$. *Differentiation of dyadics.*

§ 21. *The dyadic* $\square\mathfrak{s}$. In addition to the scalar and vector products of the vector operator $\square$ and a four-vector $\mathfrak{s}$ considered above, we may form the *open* product $\square\mathfrak{s}$, which is the dyadic

$$\square\mathfrak{s} = \mathfrak{i}\frac{\partial\mathfrak{s}}{\partial x} + \mathfrak{j}\frac{\partial\mathfrak{s}}{\partial y} + \mathfrak{k}\frac{\partial\mathfrak{s}}{\partial z} + \mathfrak{h}\frac{\partial\mathfrak{s}}{\partial l} \qquad (1),$$

and similarly the conjugate dyadic

$$\mathfrak{s}\square = \frac{\partial\mathfrak{s}}{\partial x}\mathfrak{i} + \frac{\partial\mathfrak{s}}{\partial y}\mathfrak{j} + \frac{\partial\mathfrak{s}}{\partial z}\mathfrak{k} + \frac{\partial\mathfrak{s}}{\partial l}\mathfrak{h} \qquad (2).$$

And each of these is a four-dyadic, because $\mathfrak{s}$ is a four-vector and $\square$ is transformed in the same way as a four-vector. The dyadics so defined are analogous to the dyadics $\nabla\mathbf{s}$ and $\mathbf{s}\nabla$ of three dimensions, and will be found equally useful.

If $\hat{\mathfrak{a}} = (a_1, a_2, a_3, a_4)$ is a unit four-vector the product

$$\hat{\mathfrak{a}} \cdot (\square\mathfrak{s}) = a_1\frac{\partial\mathfrak{s}}{\partial x} + a_2\frac{\partial\mathfrak{s}}{\partial y} + a_3\frac{\partial\mathfrak{s}}{\partial z} + a_4\frac{\partial\mathfrak{s}}{\partial l} = (\hat{\mathfrak{a}} \cdot \square)\,\mathfrak{s} \qquad (3)$$

is a four-vector representing the *directional derivative* of $\mathfrak{s}$ for the direction of $\hat{\mathfrak{a}}$. It will be noticed that this result is of the same form as (4) of § 16 in the case of a scalar function. And further if $\mathfrak{s}$ and $\mathfrak{s} + d\mathfrak{s}$ are the values of a vector function at the points whose position vectors are $\mathfrak{r}$ and $\mathfrak{r} + d\mathfrak{r}$, we have

$$
\begin{aligned}
d\mathfrak{r} \cdot \square\mathfrak{s} &= (dx\,\mathfrak{i} + dy\,\mathfrak{j} + \ldots) \cdot \left(\mathfrak{i}\frac{\partial\mathfrak{s}}{\partial x} + \mathfrak{j}\frac{\partial\mathfrak{s}}{\partial y} + \ldots\right) \\
&= \frac{\partial\mathfrak{s}}{\partial x}dx + \frac{\partial\mathfrak{s}}{\partial y}dy + \ldots + \frac{\partial\mathfrak{s}}{\partial l}dl \\
&= d\mathfrak{s} \qquad\qquad (4),
\end{aligned}
$$

which corresponds to the formula

$$d\mathfrak{r}.\Box\phi = d\phi$$

found in § 16.

We may also remark that the scalar of the dyadic $\Box\mathfrak{s}$ is

$$(\Box\mathfrak{s})_s = \mathfrak{i}\cdot\frac{\partial\mathfrak{s}}{\partial x} + \mathfrak{j}\cdot\frac{\partial\mathfrak{s}}{\partial y} + \dots$$

$$= \Box\cdot\mathfrak{s} = \operatorname{div}\mathfrak{s} \quad \dots\dots\dots\dots(5).$$

Similarly its vector is

$$(\Box\mathfrak{s})_v = \mathfrak{i}\times\frac{\partial\mathfrak{s}}{\partial x} + \mathfrak{j}\times\frac{\partial\mathfrak{s}}{\partial y} + \dots$$

$$= \Box\times\mathfrak{s} = \operatorname{curl}\mathfrak{s} \quad \dots\dots\dots\dots(6).$$

The scalar of $\mathfrak{s}\,\Box$ is also div $\mathfrak{s}$, but its vector is $-$ curl $\mathfrak{s}$. Consistently with the previous notation we may write

$$\mathfrak{s}\cdot\Box = \Sigma\frac{\partial\mathfrak{s}}{\partial x}\cdot\mathfrak{i} = \operatorname{div}\mathfrak{s},$$

$$\mathfrak{s}\times\Box = \Sigma\frac{\partial\mathfrak{s}}{\partial x}\times\mathfrak{i} = -\operatorname{curl}\mathfrak{s}.$$

It should be observed too that the six-vector $\Box\times\mathfrak{s}$ corresponds to the anti-selfconjugate dyadic $(\Box\mathfrak{s} - \mathfrak{s}\Box)$, and is equal to half its vector (§ 11). Thus if $\mathfrak{b}$ is any four-vector

$$\mathfrak{b}\times(\Box\times\mathfrak{s}) = -\,\mathfrak{b}\cdot(\Box\mathfrak{s} - \mathfrak{s}\Box) \quad \dots\dots\dots\dots(7).$$

If ϕ is a scalar point function $\Box\phi$ is a four-vector, and $\Box\Box\phi$ is a four-dyadic whose vector is $\Box\times\Box\phi$, which vanishes identically by § 18. Since then its vector is identically zero, the dyadic $\Box\Box\phi$ is selfconjugate.

Ex. Shew that $\Box\mathfrak{r} = \mathrm{I}$, where $\mathfrak{r}$ has the usual meaning.

§ 22. *Differentiation of dyadics.* If $\mathfrak{a}\mathfrak{b}$ is a dyad of four-vector point functions, we define its derivative with respect to x by the equation

$$\frac{\partial}{\partial x}(\mathfrak{a}\,\mathfrak{b}) = \frac{\partial\mathfrak{a}}{\partial x}\,\mathfrak{b} + \mathfrak{a}\,\frac{\partial\mathfrak{b}}{\partial x},$$

the open product being differentiated by the same rule as other products. Similarly if Φ is a dyadic whose antecedents and consequents are four-vector point functions, we define $\frac{\partial}{\partial x}\Phi$ as the sum of the derivatives of its dyads.

Forming then the scalar product of $\Box$ and the dyadic Φ, we have

$$\Box\cdot\Phi = \mathfrak{i}\cdot\frac{\partial\Phi}{\partial x} + \mathfrak{j}\cdot\frac{\partial\Phi}{\partial y} + \dots + \mathfrak{h}\cdot\frac{\partial\Phi}{\partial l} \quad \dots\dots\dots(8),$$

a four-vector which Minkowski denoted by $-$ lor Φ. We have already met a particular case of this, viz. lor F, when the dyadic Φ was anti-selfconjugate. Similarly we should attach to $\Phi \cdot \Box$ the meaning

$$\Phi \cdot \Box = \Sigma \frac{\partial \Phi}{\partial x} \cdot \mathfrak{i} = \Sigma\, \mathfrak{i} \cdot \frac{\partial \Phi_c}{\partial x} = \Box \cdot \Phi_c \quad \ldots\ldots\ldots (8'),$$

the dyadic Φ used as a prefactor to $\Box$ being equivalent to its conjugate Φ_c used as a postfactor. Finally we could define the function

$$\Box \times \Phi = \mathfrak{i} \times \frac{\partial \Phi}{\partial x} + \mathfrak{j} \times \frac{\partial \Phi}{\partial y} + \ldots + \mathfrak{h} \times \frac{\partial \Phi}{\partial l} \quad \ldots\ldots\ldots (9),$$

which is a dyadic whose antecedents are six-vectors and whose consequents are four-vectors. Such mixed dyadics have their use in the vector analysis of Relativity, especially in General Relativity; but a discussion of their properties will not be entered into in the present paper.

There are also two important symbolic products in which $\Box$ occurs twice in association with a four-vector $\mathfrak{s}$. Thus

$$\begin{aligned} \Box \cdot (\Box \mathfrak{s}) &= \Box \cdot \left(\mathfrak{i} \frac{\partial \mathfrak{s}}{\partial x} + \mathfrak{j} \frac{\partial \mathfrak{s}}{\partial y} + \ldots + \mathfrak{h} \frac{\partial \mathfrak{s}}{\partial l} \right) \\ &= \frac{\partial^2 \mathfrak{s}}{\partial x^2} + \frac{\partial^2 \mathfrak{s}}{\partial y^2} + \frac{\partial^2 \mathfrak{s}}{\partial z^2} + \frac{\partial^2 \mathfrak{s}}{\partial l^2} \\ &= \Box^2 \mathfrak{s} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10), \end{aligned}$$

which should be compared with (13) of § 18. Similarly

$$\begin{aligned} \Box \cdot (\mathfrak{s} \Box) &= \Sigma\, \mathfrak{i} \cdot \frac{\partial}{\partial x} \left(\frac{\partial \mathfrak{s}}{\partial x} \mathfrak{i} + \frac{\partial \mathfrak{s}}{\partial y} \mathfrak{j} + \ldots \right) \\ &= \Sigma\, \mathfrak{i} \frac{\partial}{\partial x} \left(\Sigma\, \mathfrak{i} \cdot \frac{\partial \mathfrak{s}}{\partial x} \right) \\ &= \Box\, (\Box \cdot \mathfrak{s}) = \text{grad div } \mathfrak{s} \quad \ldots\ldots\ldots\ldots (11). \end{aligned}$$

§ 23. *Formulæ of expansion.* Various useful formulæ, in some degree analogous to those of § 19, will now be proved. Consider first the formulæ

$$\begin{aligned} \Box\, (\mathfrak{a} \cdot \mathfrak{b}) &= \Box \mathfrak{a} \cdot \mathfrak{b} + \Box \mathfrak{b} \cdot \mathfrak{a} \quad \ldots\ldots\ldots\ldots (12), \\ \Box \cdot (\mathfrak{a}\, \mathfrak{b}) &= \Box \cdot \mathfrak{a}\, \mathfrak{b} + \mathfrak{a} \cdot \Box \mathfrak{b} \quad \ldots\ldots\ldots\ldots (13), \\ \Box \times (\mathfrak{a} \times \mathfrak{b}) &= \Box \cdot (\mathfrak{b}\, \mathfrak{a} - \mathfrak{a}\, \mathfrak{b}) \quad \ldots\ldots\ldots\ldots (14), \end{aligned}$$

in which $\square$ is to be understood as operating only on the vector next to it, unless brackets are used to indicate the contrary. The first and third of these are alternative to (24) and (23) of § 19. To prove (12) we may write

$$\square(\mathfrak{a}\cdot\mathfrak{b}) = \Sigma\,\mathfrak{i}\left(\frac{\partial\mathfrak{a}}{\partial x}\cdot\mathfrak{b} + \mathfrak{a}\cdot\frac{\partial\mathfrak{b}}{\partial x}\right)$$

$$= \left(\Sigma\,\mathfrak{i}\frac{\partial\mathfrak{a}}{\partial x}\right)\cdot\mathfrak{b} + \left(\Sigma\,\mathfrak{i}\frac{\partial\mathfrak{b}}{\partial x}\right)\cdot\mathfrak{a}$$

$$= \square\mathfrak{a}\cdot\mathfrak{b} + \square\mathfrak{b}\cdot\mathfrak{a}.$$

Similarly in the case of (13)

$$\square\cdot(\mathfrak{a}\,\mathfrak{b}) = \Sigma\,\mathfrak{i}\cdot\left(\frac{\partial\mathfrak{a}}{\partial x}\,\mathfrak{b} + \mathfrak{a}\,\frac{\partial\mathfrak{b}}{\partial x}\right)$$

$$= \left(\Sigma\,\mathfrak{i}\cdot\frac{\partial\mathfrak{a}}{\partial x}\right)\mathfrak{b} + a\cdot\left(\Sigma\,\mathfrak{i}\frac{\partial\mathfrak{b}}{\partial x}\right)$$

$$= \square\cdot\mathfrak{a}\,\mathfrak{b} + \mathfrak{a}\cdot\square\mathfrak{b},$$

as required. Using this result we may write the expansion for $\square\times(\mathfrak{a}\times\mathfrak{b})$, found in § 19, as

$$\square\times(\mathfrak{a}\times\mathfrak{b}) = \square\cdot(\mathfrak{b}\,\mathfrak{a} - \mathfrak{a}\,\mathfrak{b}),$$

thus providing (14). This formula is also obvious from the fact that $\mathfrak{a}\times\mathfrak{b}$ corresponds to the anti-selfconjugate dyadic $\mathfrak{a}\,\mathfrak{b} - \mathfrak{b}\,\mathfrak{a}$, and therefore

$$\square\times(\mathfrak{a}\times\mathfrak{b}) = -\,\square\cdot(\mathfrak{a}\,\mathfrak{b} - \mathfrak{b}\,\mathfrak{a}),$$

which is the same as (14).

Again if u, $\mathfrak{a}$, Φ are point functions—scalar, four-vector and dyadic respectively—the following formulæ hold:

$$\square(u\,\mathfrak{a}) \;= \square\,u\,\mathfrak{a} + u\,\square\,\mathfrak{a} \quad \text{..............(15)},$$

$$\square\cdot(u\Phi) = \square\,u\cdot\Phi + u\,\square\cdot\Phi \quad \text{...........(16)}.$$

For in the case of the second of these

$$\square\cdot(u\Phi) = \Sigma\,\mathfrak{i}\cdot\left(\frac{\partial u}{\partial x}\,\Phi + u\,\frac{\partial\Phi}{\partial x}\right)$$

$$= \left(\Sigma\,\mathfrak{i}\frac{\partial u}{\partial x}\right)\cdot\Phi + u\,\Sigma\,\mathfrak{i}\cdot\frac{\partial\Phi}{\partial x}$$

$$= \square\,u\cdot\Phi + u\square\cdot\Phi,$$

and (15) may be proved in like manner. The corresponding formula

$$\Box \times (u\Phi) = \Box u \times \Phi + u \Box \times \Phi \quad \text{...........(17)}$$

also holds; but this again is an equation of mixed dyadics, with which we have decided not to deal. As a particular case of (16) we may notice

$$\Box \cdot (u \mathrm{I}) = \Box u \quad \text{....................(18).}$$

Lastly we may easily expand the expressions

$$\Box \cdot (\Phi \cdot \mathfrak{a}), \quad \Box \times (\Phi \cdot \mathfrak{a}),$$

where Φ and $\mathfrak{a}$ are point functions. For

$$\Box \cdot (\Phi \cdot \mathfrak{a}) = \Sigma \mathfrak{i} \cdot \left(\frac{\partial \Phi}{\partial x} \cdot \mathfrak{a} + \Phi \cdot \frac{\partial \mathfrak{a}}{\partial x}\right)$$

$$= \left(\Sigma \mathfrak{i} \cdot \frac{\partial \Phi}{\partial x}\right) \cdot \mathfrak{a} + \left[\Phi_c \cdot \left(\Sigma \mathfrak{i} \frac{\partial \mathfrak{a}}{\partial x}\right)\right]_s$$

$$= (\Box \cdot \Phi) \cdot \mathfrak{a} + (\Phi_c \cdot \Box \mathfrak{a})_s \quad \text{.....................(19).}$$

The last term, which is the scalar of the product of Φ_c and $\Box \mathfrak{a}$, may equally well be written $(\Box \mathfrak{a} \cdot \Phi_c)_s$. Similarly

$$\Box \times (\Phi \cdot \mathfrak{a}) = \Sigma \mathfrak{i} \times \left(\frac{\partial \Phi}{\partial x} \cdot \mathfrak{a} + \Phi \cdot \frac{\partial \mathfrak{a}}{\partial x}\right)$$

$$= (\Box \times \Phi) \cdot \mathfrak{a} + (\Box \mathfrak{a} \cdot \Phi_c)_v \quad \text{...........(20).}$$

In the first term of the second member $\Box \times \Phi$ is again a mixed dyadic whose antecedents are six-vectors and whose consequents are four-vectors. But in scalar multiplication with $\mathfrak{a}$ it yields a six-vector, as required.

In closing this paper we may point out that

$$\Box \cdot (\Phi \times \mathfrak{a}) = \Sigma \mathfrak{i} \cdot \left(\frac{\partial \Phi}{\partial x} \times \mathfrak{a} + \Phi \times \frac{\partial \mathfrak{a}}{\partial x}\right)$$

$$= (\Box \cdot \Phi) \times \mathfrak{a} + (\Phi_c \cdot \Box \mathfrak{a})_v \quad \text{..........(21),}$$

each term being a six-vector. Also that, if the dyadic Φ is *constant*,

$$\Box (\Phi \cdot \mathfrak{a}) = \Sigma \mathfrak{i} \left(\Phi \cdot \frac{\partial \mathfrak{a}}{\partial x}\right) = \Box \mathfrak{a} \cdot \Phi_c \quad \text{..........(22).}$$

ON A DIOPHANTINE PROBLEM.

(THIRD PAPER).

By *H. Holden.*

1. IN a previous paper* Legendre's system of equations was solved by writing them as

$$x^2 + y^2 = 2c^2,$$
$$y^2 + z^2 = 2a^2,$$
$$z^2 + x^2 = 2b^2.$$

Using for the first two equations the linear relations

$$(k^2 - 2k - 1)x - (k^2 + 2k - 1)y = 0,$$
$$(m^2 - 2m - 1)y - (m^2 + 2m - 1)z = 0,$$

it was pointed out that smaller solutions would probably be got if $k^2 - 2k - 1$ and $m^2 + 2m - 1$ could be made equal. A method for doing so was shown, but it would have been simpler to have put $k = m + 2$, when we get

$$x = m^2 + 6m + 7,$$
$$y = m^2 + 2m - 1,$$
$$z = m^2 - 2m - 1,$$

and $$b^2 = m^4 + 4m^3 + 26m^2 + 44m + 25$$
$$= (m^2 + \tfrac{22}{5}m + 5)^2 \text{ if } m = -\tfrac{7}{10},$$

which gives
$$x = 329, \qquad a = 149,$$
$$y = 191, \qquad b = 241,$$
$$z = 89, \qquad c = 269,$$

as before. This method may be used for any number of equations of the above type.

Thus for the system

$$x_1^2 + x_2^2 = 2a_1^2,$$
$$x_2^2 + x_3^2 = 2a_2^2,$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$x_{p-1}^2 + x_p^2 = 2a_{p-1}^2,$$
$$x_p^2 + x_1^2 = 2a_p^2,$$

* Vol. xlviii., p. 85.

we may take for x_1 to x_p any p consecutive terms of the series whose general term is $(m+2r+1)^2-2$. The values of a_1 to a_{p-1} will then be consecutive terms of the series whose general term is $(m+2r)^2+1$, and it only remains to find a value of m which will satisfy the last equation. This can always be done.

Thus if $p=2n$, taking for x_1 to x_{2n} the terms ranging from $(m-2n+1)^2-2$ to $(m+2n-1)^2-2$, the system will be solved if

$$m=\frac{2n(2n-1)(2n^2-2n-1)}{2n^2-2n+1}.$$

If $p=2n+1$, and if we take for x_1 to x_{2n+1} the terms ranging from $(m-2n+1)^2-2$ to $(m+2n+1)^2-2$, we get

$$m=\frac{16n^4-24n^2+1}{2(4n^2+1)}$$

and
$$a_{2n+1}=m^2+\frac{24n^2-2}{4n^2+1}.m+4n^2+1.$$

Thus for the five equations

$$\begin{aligned}
x_1^2+x_2^2&=2a_1^2,\\
x_2^2+x_3^2&=2a_2^2,\\
x_3^2+x_4^2&=2a_3^2,\\
x_4^2+x_5^2&=2a_4^2,\\
x_5^2+x_1^2&=2a_5^2,
\end{aligned}$$

take
$$\begin{aligned}
x_1&=m^2+10m+23, & a_1&=m^2+8m+17,\\
x_2&=m^2+\ \ 6m+\ \ 7, & a_2&=m^2+4m+\ \ 5,\\
x_3&=m^2+\ \ 2m-\ \ 1, & a_3&=m^2+1,\\
x_4&=m^2-\ \ 2m-\ \ 1, & a_4&=m^2-4m+5,\\
x_5&=m^2-\ \ 6m+\ \ 7,
\end{aligned}$$

and
$$\begin{aligned}
a_5^2&=m^4+4m^3+98m^2+188m+289\\
&=\left(m^2+\frac{94m}{17}+17\right)^2 \text{ if } m=\tfrac{161}{34}.
\end{aligned}$$

The above equations may be written in either of the forms

$$\begin{aligned}
a_1^2-a_2^2+a_3^2-a_4^2+a_5^2&=x_1^2,\\
a_2^2-a_3^2+a_4^2-a_5^2+a_1^2&=x_2^2,\\
a_3^2-a_4^2+a_5^2-a_1^2+a_2^2&=x_3^2,\\
a_4^2-a_5^2+a_1^2-a_2^2+a_3^2&=x_4^2,\\
a_5^2-a_1^2+a_2^2-a_3^2+a_4^2&=x_5^2,
\end{aligned}$$

and
$$a_1^2+a_2^2+a_3^2-a_4^2+a_5^2=x_1^2+x_2^2+x_3^2,$$
$$a_2^2+a_3^2+a_4^2-a_5^2+a_1^2=x_2^2+x_3^2+x_4^2,$$
$$a_3^2+a_4^2+a_5^2-a_1^2+a_2^2=x_3^2+x_4^2+x_5^2,$$
$$a_4^2+a_5^2+a_1^2-a_2^2+a_3^2=x_4^2+x_5^2+x_1^2,$$
$$a_5^2+a_1^2+a_2^2-a_3^2+a_4^2=x_5^2+x_1^2+x_2^2,$$
so that these, and similar systems containing any odd number of equations, are satisfied by the solutions found above.

2. Apparently, in order to obtain a linear relation, it must be possible to arrange the terms of an equation in two groups, each of the same quadratic form.

The system
$$a^2-2sab+tb^2+(s^2-t)c^2=z^2,$$
$$b^2-2sbc+tc^2+(s^2-t)a^2=x^2,$$
$$c^2-2sca+ta^2+(s^2-t)b^2=y^2,$$
where s and t are any given rational quantities, satisfies this condition.

If $t=1$, solutions can be found which are expressed in terms of one rational parameter; otherwise numerical solutions can be obtained. To prove this statement write the first equation as
$$(a-sb)^2-(s^2-t)b^2+(s^2-t)c^2=z^2,$$
so that its linear relation is
$$2k(a-sb)-(k^2+s^2-t)b+(k^2-s^2+t)c=0,$$
or
$$2ka-(k^2+2ks+s^2-t)b+(k^2-s^2+t)c=0.$$
Similarly, for the second equation
$$2mb-(m^2+2ms+s^2-t)c+(m^2-s^2+t)a=0.$$

Let the value of a found from these two relations be $a=p_1k^2+q_1k+r_1$, with similar expressions for b and c. Substituting in the first relation we have $p_2=p_3$ and $r_2=-r_3$, and a similar substitution in the third equation shews that the coefficient of k^4 will be
$$p_3^2-2sp_3p_1+tp_1^2+(s^2-t)p_2^2$$
$$=(s^2-t+1)p_3^2-2sp_3p_1+tp_1^2$$
$$=(sp_3-p_1)^2 \text{ if } t=1;$$
and similarly the term independent of k will be $(sr_3-r_1)^2$, so that suitable values of k, expressed in terms of m, may be obtained.

If t is not equal to 1, let

$$a = k^2 (p_1 m^2 + \ldots + \ldots) + \ldots + \ldots,$$

with similar expressions for b and c.

The first linear relation gives $p_2 = p_3$, and the second shew that $p_3 = p_1$, and so, on substituting in the third equation, the coefficient of k^4 will have, as its first term,

$$m^4 \{p_1^{\,2} - 2sp_1^{\,2} + tp_1^{\,2} + (s^2 - t)\, p_1^{\,2}\} = m^4 p_1^{\,2} (s-1)^2;$$

and so, in general, values of m may be found which make this coefficient a square, and from this values of k may be got which make the biquadratic expression a square.

As an example, take the system

$$a^2 + 4ab + 7b^2 - 3c^2 = z^2,$$
$$b^2 + 4bc + 7c^2 - 3a^2 = x^2,$$
$$c^2 + 4ca + 7a^2 - 3b^2 = y^2.$$

The linear relations

$$2ka + (k^2 + 4k - 3)\, b - (k^2 + 3)\, c = 0$$

and

$$-2a + b + c = 0$$

give

$$a = k^2 + 2k,$$
$$b = k^2 - k + 3,$$
$$c = k^2 + 5k - 3,$$

and

$$y^2 = 9\,(k^4 + 8k^3 + 6k^2 - 4k - 2)$$
$$= 9\,(k^2 + 4k - 5)^2 \text{ if } k = \tfrac{3}{4}.$$

Hence

$$a = 11, \qquad x = 25,$$
$$b = 15, \qquad y = 23,$$
$$c = 7, \qquad z = 47.$$

These values also satisfy a fourth equation

$$c^2 + 4bc + 7b^2 - 3a^2 = x_1^{\,2}.$$

Or, take the system

$$4a^2 - 4ab - 2b^2 + 3c^2 = z^2,$$
$$4b^2 - 4bc - 2c^2 + 3a^2 = x^2,$$
$$4c^2 - 4ca - 2a^2 + 3b^2 = y^2.$$

The linear relations

$$4ka - (k^2 + 2k + 3)\, b + (k^2 - 3)\, c = 0$$

and
$$a+2b-3c=0$$
give
$$a=k^2+6k+15,$$
$$b=k^2+12k-3,$$
$$c=k^2+10k+3,$$
and
$$y^2=k^4+64k^3+394k^2-1008k-567$$
$$=(k^2+32k-315)^2 \text{ if } k=\tfrac{99}{19}.$$
Hence
$$a=631, \qquad x=555,$$
$$b=745, \qquad y=1041,$$
$$c=707, \qquad z=319.$$

As an example, where a more general solution may be obtained, consider the system
$$4(a^2+ab+b^2)-3c^2=z^2,$$
$$4(b^2+bc+c^2)-3a^2=x^2,$$
$$4(c^2+ca+a^2)-3b^2=y^2.$$
Writing the equations in the form
$$(2a+b)^2+3b^2-3c^2=z^2,$$
we have
$$4ka+(k^2+2k-3)b-(k^2+3)c=0,$$
$$4mb+(m^2+2m-3)c-(m^2+3)a=0;$$
and so
$$a=k^2(m^2+6m-3)+2k(m^2+2m-3)-3(m^2-2m-3),$$
$$b=k^2(m^2+3)-4k(m^2+2m-3)+3(m^2+3),$$
$$c=k^2(m^2+3)+2k(m^2+8m+3)-3(m^2+3),$$
and
$$y^2=9\,[k^4(m^2+4m-1)^2+8k^3(m^2+4m-1)(m+1)^2$$
$$+k^2(-10m^4+16m^3+180m^2+112m-74)$$
$$-2k(4m^4+24m^3-32m^2+8m+60)+(3m^2-4m-3)^2]$$
$$=9\,[k^2(m^2+4m-1)+4k(m+1)^2-(3m^2-4m-3)]^2,$$
if
$$k=\frac{4(m+1)(m-3)(m^2+3)}{5m^4+8m^3-10m^2-8m+21}.$$
Other solutions in general form may be obtained, or to obtain a numerical solution with less labour put $m=2$ in the expression for y^2, and we get
$$y^2=9(121k^4+792k^3+838k^2-408k+1)$$
$$=9(11k^2-204k+1)^2 \text{ if } k=\tfrac{85}{11},$$

which gives
$$a = 2899, \quad x = 3701,$$
$$b = 956, \quad y = 9262,$$
$$c = 2529, \quad z = 5401.$$

Again

$$y^2 = 9[k^2(m^2 + 4m - 1) + 4k(m+1)^2 + 3m^2 - 4m - 3]^2 \text{ if } k = -1.$$

But this value of k leads to solutions which satisfy $a + b + c = 0$, and it will be found that any values of a, b, c which satisfy this last equation are also solutions of the system.

These simpler but partial solutions, in cases giving positive values of a, b, c, may now be considered. They may be obtained when $t = 1$ or $2s$.

3. The first two equations of the system

$$a^2 - 2sab + b^2 + (s^2 - 1)c^2 = z^2,$$
$$b^2 - 2sbc + c^2 + (s^2 - 1)a^2 = x^2,$$
$$c^2 - 2sca + a^2 + (s^2 - 1)b^2 = y^2,$$

are satisfied by $c + a - 2sb = 0$ and the third equation by

$$2kc - (k^2 + 2ks + s^2 - 1)a \pm (k^2 - s^2 + 1)b = 0.$$

The upper sign yields

$$a = k^2 + 4ks - s^2 + 1,$$
$$b = k^2 + 2k(s+1) + s^2 - 1,$$
$$c = k^2(2s - 1) + 4ks^2 + (s^2 - 1)(2s + 1),$$

and the lower sign gives

$$a = -k^2 + 4ks + s^2 - 1,$$
$$b = k^2 + 2k(s+1) + s^2 - 1,$$
$$c = k^2(2s + 1) + 4ks^2 + (2s - 1)(s^2 - 1).$$

For positive solutions s must be positive. Again, the first two equations of the system

$$a^2 - 2sab + b^2 - c^2 = (1 - s^2)z^2,$$
$$b^2 - 2sbc + c^2 - a^2 = (1 - s^2)x^2,$$
$$c^2 - 2sca + a^2 - b^2 = (1 - s^2)y^2$$

are satisfied by $c + a - sb = 0$, and the third equation by

$$(k^2 + s^2 - 1)c - \{k^2s + 2k(s^2 - 1) + s(s^2 - 1)\}a \pm (k^2 - s^2 + 1)b = 0,$$

where, again, s should be positive. The upper sign gives

$$a = k^2 + (s-1)^2,$$
$$b = k^2 + 2k(s-1) + s^2 - 1,$$
$$c = k^2(s-1) + 2ks(s-1) + (s-1)(s^2+1),$$

and the lower sign

$$a = k^2(s-1) + (s+1)(s^2-1),$$
$$b = k^2(s+1) + 2k(s^2-1) + (s+1)(s^2-1),$$
$$c = k^2(s^2+1) + 2ks(s^2-1) + (s^2-1)^2.$$

Lastly, the first two equations of the system

$$a^2 - 2sab + 2sb^2 + (s^2-2s)c^2 = z^2,$$
$$b^2 - 2sbc + 2sc^2 + (s^2-2s)a^2 = x^2,$$
$$c^2 - 2sca + 2sa^2 + (s^2-2s)b^2 = y^2$$

are satisfied by $c + a - b = 0$, and the third equation by

$$2kc - (k^2 + 2ks + s^2 - 2s)a \pm (k^2 - s^2 + 2s)b = 0.$$

Taking the upper sign

$$a = k^2 + 2k - s^2 + 2s,$$
$$b = k^2 + 2k(s+1) + s^2 - 2s,$$
$$c = 2ks + 2(s^2 - 2s),$$

and with the lower sign we get

$$a = - k^2 + 2k + s^2 - 2s,$$
$$b = k^2 + 2k(s+1) + s^2 - 2s,$$
$$c = 2k^2 + 2ks.$$

4. Solutions, expressed in terms of one or two parameters, may be obtained of the system

$$a^2 - 2sab + b^2 + (s^2-1)c^2 = z^2,$$
$$b^2 - 2sbc + c^2 + (s^2-1)d^2 = t^2,$$
$$c^2 - 2scd + d^2 + (s^2-1)a^2 = x^2,$$
$$d^2 - 2sda + a^2 + (s^2-1)b^2 = y^2.$$

The linear relation for the second equation should be written as

$$(k^2 - 2ks + s^2 - 1)b + 2kc - (k^2 - s^2 + 1)d = 0,$$

and, on expressing the values of a, b, c, d in powers of k according to the previous notation, we find that $p_2 = p_4$. On substituting these values of a, b, c, d in the last equation, the coefficient of k^4 in the expression for y^2 will be

$$p_4^2 - 2sp_4p_1 + p_1^2 + (s^2-1)p_2^2 = (p_1 - sp_4)^2,$$

and so values of k, expressed in terms of the other two parameters, may be found. Thus for the system

$$\begin{aligned} 4(a^2 - ab + b^2) - 3c^2 &= z^2, \\ 4(b^2 - bc + c^2) - 3d^2 &= t^2, \\ 4(c^2 - cd + d^2) - 3a^2 &= x^2, \\ 4(d^2 - da + a^2) - 3b^2 &= y^2, \end{aligned}$$

we may get special solutions by using the relations

$$\begin{gathered} -a + b - c = 0, \\ (k^2 - 2k - 3)\,b + 4kc - (k^2 + 3)\,d = 0, \\ -3c + 8d - 7a = 0. \end{gathered}$$

These yield

$$\begin{aligned} a &= 5k^2 + 16k - 33, \\ b &= 4k^2 + 32k + 12, \\ c &= -k^2 + 16k + 45, \\ d &= 4k^2 + 20k - 12, \end{aligned}$$

and

$$\begin{aligned} y^2 &= 36k^4 - 144k^3 - 2952k^2 - 5040k + 2916 \\ &= (6k^2 - 12k - 258)^2 \text{ if } k = -\tfrac{17}{3} \\ &= (6k^2 - 12k - 54)^2 \text{ if } k = -\tfrac{44}{17}. \end{aligned}$$

The value $k = -\frac{17}{3}$ gives

$$\begin{aligned} a &= 83, & x &= 327, \\ b &= -92, & y &= 6, \\ c &= -175, & z &= 9, \\ d &= 7, & t &= 303, \end{aligned}$$

whilst $k = -\frac{44}{17}$ yields

$$\begin{aligned} a &= 11825, & x &= 1383, \\ b &= 12724, & y &= 4986, \\ c &= 899, & z &= 24549, \\ d &= 10684, & t &= 16206. \end{aligned}$$

Had we used the relations

$$\begin{aligned} -a+b-c&=0, \\ (k^2-2k-3)\,b+4kc-(k^2+3)\,d&=0, \\ 3c+5d-7a&=0, \end{aligned}$$

we should have got

$$\begin{aligned} a&=4k^2+\ \ 5k-\ \ 3, \\ b&=5k^2+10k+15, \\ c&=\ \ k^2+\ \ 5k+18, \\ d&=5k^2+\ \ 4k-15, \end{aligned}$$

and
$$\begin{aligned} y^2&=9\,(k^4-16k^3-118k^2-128k+9) \\ &=9\left(k^2+\frac{64k}{3}-3\right)^2 \text{ if } k=-\tfrac{29}{3} \\ &=9\,(k^2-8k-91)^2 \quad \text{if } k=-\tfrac{47}{9}. \end{aligned}$$

The value $k=-\frac{29}{3}$ gives

$$\begin{aligned} a&=1451, & x&=2397, \\ b&=1735, & y&=1563, \\ c&=\ \ 284, & z&=3186, \\ d&=1861, & t&=\ \ \ 51, \end{aligned}$$

and from $k=-\frac{47}{9}$*

$$\begin{aligned} a&=3239, & x&=4953, \\ b&=4015, & y&=2667, \\ c&=\ \ 776, & z&=7254, \\ d&=4069, & t&=2181. \end{aligned}$$

5. For the system

$$\begin{aligned} (a+b)^2+b^2-(b+c)^2&=y^2, \\ (b+c)^2+c^2-(c+a)^2&=z^2, \\ (c+a)^2+a^2-(a+b)^2&=x^2 \end{aligned}$$

* $k=-\frac{13}{3}$ gives

$$\begin{aligned} a&=227, & x&=309, \\ b&=295, & y&=\ \ 21, \\ c&=\ \ 68, & z&=522, \\ d&=277, & t&=237. \end{aligned}$$

use $$(k^2-1)\,a+(2k-2)\,b-(k^2+1)\,c=0,$$
$$-5a+3b+2c=0,$$
which give
$$a=3k^2+4k-1,$$
$$b=3k^2+7,$$
$$c=3k^2+10k-13,$$
and
$$x^2=9k^4+144k^3-50k^2-448k+161$$
$$=(3k^2+24k-\tfrac{313}{3})^2 \text{ if } k=\tfrac{127}{54}.$$
Hence
$$a=1279,\qquad x=1601,$$
$$b=1207,\qquad y=953,$$
$$c=1387,\qquad z=1243.$$

Again, for the system
$$bc+ca-ab=z^2,$$
$$cd+db-bc=t^2,$$
$$da+ac-cd=x^2,$$
$$ab+bd-da=y^2,$$
take
$$a+4b-c=0,$$
$$k^2b+c-(k+1)^2\,d=0,$$
$$c+4d-9a=0,$$
whence
$$a=2k^2+2k+5,$$
$$b=2k^2+4k+1,$$
$$c=10k^2+18k+9,$$
$$d=2k^2+9.$$
with
$$y^2=4k^4+16k^3+12k^2+40k-31$$
$$=(2k^2+4k-1)^2 \text{ if } k=\tfrac{2}{3};$$
and so
$$a=65,\qquad x=17,$$
$$b=41,\qquad y=23,$$
$$c=229,\qquad z=147,$$
$$d=89,\qquad t=121.$$

A solution expressed in terms of two parameters may be obtained.

As another example, consider the system

$$sab + c^2 = z^2,$$
$$sbc + a^2 = x^2,$$
$$sca + b^2 = y^2,$$

where s is a given rational quantity. A solution in general form may be obtained; otherwise for the first two equations use

$$4a - sb + 4c = 0,$$

and for the third equation

$$4k^2a + 4kb - sc = 0,$$

which gives
$$\begin{aligned} a &= -16k + s^2, \\ b &= 16k^2 + 4s, \\ c &= 4k^2s + 16k. \end{aligned}$$

Thus, if $s = 1$,
$$\begin{aligned} a &= 9, \\ b &= 328, \\ c &= 73, \end{aligned}$$

and for $s = 4$
$$\begin{aligned} a &= 3, \\ b &= 5, \\ c &= 2. \end{aligned}$$

If positive solutions be desired, s must be positive.

It may be noticed that the systems

$$ab \pm 1 = z^2,$$
$$bc \pm 1 = x^2,$$
$$ca \pm 1 = y^2$$

may be solved by taking for a, b, c, with the upper sign three consecutive terms of even ranks, and with the lower sign, three consecutive terms of odd rank, of Fibonacci's series.

6. Hitherto, each equation of the system has been of the same form, but the method applies equally well when this is not the case. Thus the system

$$qa^2 + b^2 - qc^2 = z^2,$$
$$b^2 + pc^2 - pa^2 = x^2,$$
$$s(s+1)c^2 + (s+1)a^2 - n^2sb^2 = y^2,$$

where q, p, n, s are any given rational quantities, may be easily solved in general terms. Thus, for all rational values of s, the last equation is satisfied by

$$-nb+c+a=0,$$

and the second equation by

$$2kb+(k^2-p)\,c+(k^2+p)\,a=0,$$

which give

$$\begin{aligned} a &= nk^2+2k-np, \\ b &= \qquad\quad 2p, \\ c &= nk^2+2k+np, \end{aligned}$$

so that

$$z^2=4\,(-n^2pqk^2-2npqk+p^2)$$

instead of the usual biquadratic expression in k. This is satisfied by

$$k=-\frac{2p\,(m+nq)}{m^2+n^2pq},$$

where m is any rational parameter, so that

$$\begin{aligned} a &= n\,(m^2+n^2pq)^2+4m\,(m+nq)\,(m-np), \\ b &= 2\,(m^2+n^2pq)^2, \\ c &= n\,(m^2+n^2pq)^2-4m\,(m+nq)\,(m-np). \end{aligned}$$

For the special values $m=\frac{1}{2}n\,(p-q)$, $m=-np$, or $m=nq$, we get in each case

$$\begin{aligned} a &= \;\; n^2\,(p+q)^2-8\,(p-q), \\ b &= 2n\;\,(p+q)^2, \\ c &= \;\; n^2\,(p+q)^2+8\,(p-q), \end{aligned}$$

which are available, unless $p=q$. Thus, for the system

$$\begin{aligned} a^2+\;\,b^2-\;\,c^2 &= z^2, \\ b^2+2c^2-2a^2 &= x^2, \\ c^2+3a^2-3b^2 &= y^2, \end{aligned}$$

solutions are

$$\begin{aligned} a &= \;\,7, & x &= 15, \\ b &= \;\,9, & y &= \;\,5, \\ c &= 11, & z &= \;\,3. \end{aligned}$$

The same values of a, b, and c would satisfy more general equations. Thus the second equation might be written

$$b^2+\frac{p\,(p-3)}{2}\,c^2-\frac{p\,(p-3)}{2}\,a^2=x_1^{\,2}.$$

Similarly, taking the system

$$a^2 + b^2 - c^2 = z^2,$$
$$b^2 + 3c^2 - 3a^2 = x^2,$$
$$c^2 + 3a^2 - 3b^2 = y^2,$$

a solution is given by $a = 3$, $b = 4$, $c = 5$, and these values also satisfy the system

$$\begin{aligned} &a^2 + (1 - m^2)\, b^2 &&- (1 - m^2)\, c^2 &&= z_1^2, \\ &b^2 + (n^2 - 1)\, c^2 &&- (n^2 - 1)\, a^2 &&= x_1^2, \\ &c^2 + (3 + 4p - 7p^2)\, a^2 &&- (3 + 4p - 7p^2)\, b^2 &&= y_1^2. \end{aligned}$$

If we can satisfy

$$1 - m^2 = n^2 - 1 = 3 + 4p - 7p^2,$$

we can find a symmetrical system, of which solutions are $a = 3$, $b = 4$, $c = 5$.

The first equation is satisfied if

$$m = \frac{k^2 - 2k - 1}{k^2 + 1} \text{ and } n = \frac{k^2 + 2k - 1}{k^2 + 1},$$

and the second if $n^2 = 4 + 4p - 7p^2$, which requires

$$p = \frac{4(1 - x)}{x^2 + 7} \text{ and } n = \frac{14 + 4x - 2x^2}{x^2 + 7}.$$

It only remains to find rational solutions of

$$\frac{k^2 + 2k - 1}{k^2 + 1} = \frac{14 + 4x - 2x^2}{x^2 + 7}.$$

If this be expressed in powers of x, the condition for rational roots requires that a certain biquadratic expression in k should be a square, and by Fermat's method we find that a suitable value of k is $\frac{50}{7}$, which gives $n = \frac{3151}{2549}$, and so the coefficients in the symmetrical system may be obtained. The same result could of course be obtained by writing the system as

$$a^2 + pb^2 - pc^2 = z^2,$$
$$b^2 + pc^2 - pa^2 = x^2,$$
$$c^2 + pa^2 - pb^2 = y^2,$$

and on substituting $a = 3$, $b = 4$, $c = 5$ it is necessary that each of the expressions $1 - p$, $1 + p$, $25 - 7p$ should be a square. A suitable value of p can then be found by the triple equation method of Fermat.

7. The solution of the system

$$
\begin{aligned}
sa_1a_2 \quad &+ a_3^2 = x_1^2, \\
sa_2a_3 \quad &+ a_4^2 = x_2^2, \\
\vdots \qquad &\quad \vdots \qquad \vdots \\
sa_{n-1}a_n &+ a_1^2 = x^2_{n-1}, \\
sa_na_1 \quad &+ a_2^2 = x_n^2
\end{aligned}
$$

is exceptionally simple. For if the first relation be written as $4k^2a_1 - sa_2 + 4ka_3 = 0$, it will be seen, using the previous notation, that $p_1 = 0$. Or, writing the $(n-1)^{\text{th}}$ relation as $sa_{n-1} - 4k^2a_n + 4ka_1 = 0$, we get $p_n = 0$. In either case the coefficient of k^4 in $sa_na_1 + a_2^2$ will be p_2^2, so that solutions, expressed in terms of the $(n-2)$ parameters, other than k, may in general be obtained.

For the system

$$
\begin{aligned}
4ab + c^2 &= x^2, \\
4bc + d^2 &= y^2, \\
4cd + e^2 &= z^2, \\
4de + a^2 &= t^2, \\
4ea + b^2 &= u^2,
\end{aligned}
$$

we get small positive solutions by taking the relations

$$
\begin{aligned}
a - b + c &= 0, \\
b - 4c + 2d &= 0, \\
c - d + e &= 0, \\
d - k^2e + ka &= 0,
\end{aligned}
$$

from which

$$
\begin{aligned}
a &= - k^2 + 3, \\
b &= -2k^2 - 2k + 4, \\
c &= - k^2 - 2k + 1, \\
d &= - k^2 - 3k, \\
e &= - k - 1,
\end{aligned}
$$

and

$$
\begin{aligned}
u^2 &= 4k^4 + 12k^3 - 8k^2 - 28k + 4 \\
&= (2k^2 + 3k + 2)^2 \text{ if } k = -\tfrac{8}{5}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
a &= 11, & x &= 63, \\
b &= 52, & y &= 108, \\
c &= 41, & z &= 97, \\
d &= 56, & t &= 61, \\
e &= 15, & u &= 58.
\end{aligned}
$$

CIRCULAR PARTS: THE GENERAL CASE.

By *W. Woolsey Johnson.*

1. In a former paper* the writer expressed the hope of being able to make a contribution to the discussion of the general case of circular parts suggested by Napier's class in case of five circular parts.

We consider therefore n quantities $x_1, x_2, \ldots, x_n$ such that, being placed for convenience at the vertices of a regular n-gon, any two of them being given in value, the remaining $n-2$ are determined. There exist therefore a relation between the values of any triad of the n quantities, or "parts" as we shall call them, following Napier. Furthermore the parts are known to constitute a reversible cycle, so that the relation between the members of any triad depends solely upon the "collocation" of the members as they stand at the vertices of the n-gon.

2. The existence of an indefinite number of sets of circular parts is obvious geometrically. In particular it is proposed to develop the "rules" in the unique case in which the "rules" are linear.

Notations for the collocations.

3. The "parts" being placed, as above stated, at the vertices of an n-gon, and the members of a given triad marked, the collocation to which the triad belongs may be characterized by the number of vertices skipped between the nearest pairs constituting the triad. The total number of skipped vertices is $n-3$. Calling the collocation, to which the triad $x_1x_2x_3$ belongs, the "primary" one, it is characterized by the "skip numbers" 0, 0, and $n-3$. Let us write this $(0, 0, n-3)$ [or, when it is not desired to put n in evidence, $(0, 0)$. Thus for Napier's adjacent parts the symbol is $(0, 0)$ and for his opposite parts $(0, 1)$].

* *Messenger of Mathematics*, vol. xlviii. (1919), pp. 145–153.

It will be observed that for any value of n these collocation symbols are simply the partitions of $n-3$ (the number of skipped vertices) into 3 parts, admitting zeroes and repetitions.

Thus, for $n=5$, we have $\left.\begin{matrix}(0,\ 0,\ 2)\\(0,\ 1,\ 1)\end{matrix}\right\}$.

For $n=6$, $\left.\begin{matrix}(0,\ 0,\ 3)\\(0,\ 1,\ 2)\\(1,\ 1,\ 1)\end{matrix}\right\}$. For $n=7$, $\left.\begin{matrix}(0,\ 0,\ 4)\\(0,\ 1,\ 3)\\(0,\ 2,\ 2)\\(1,\ 1,\ 2)\end{matrix}\right\}$.

For $n=8$, $\left.\begin{matrix}(0,\ 0,\ 5)\\(0,\ 1,\ 4)\\(0,\ 2,\ 3)\\(1,\ 1,\ 3)\\(1,\ 2,\ 2)\end{matrix}\right\}$. For $n=9$, $\left.\begin{matrix}(0,\ 0,\ 6)\\(0,\ 1,\ 5)\\(0,\ 2,\ 4)\\(0,\ 3,\ 3)\\(1,\ 1,\ 4)\\(1,\ 2,\ 3)\\(2,\ 2,\ 2)\end{matrix}\right\}$.

The number of collocations appears to be $n-3$ until we reach $n=9$, but then and thereafter greater than $n-3$.

4. Little interest would attach to cases in which n is a composite number, although what remains to be said would apply to this case as well as when n is prime. The collocation first written, in each case, is the "primary" one. Let us suppose that for a given case the "rule" for the primary collocation, that of "adjacent parts", is known. Then the "rule" for any other collocation may be found by elimination in a manner which will be illustrated below in a unique case of circular parts, which will be now described.

The linear case of circular parts.

5. Sets of circular parts, in which the "rules" are in algebraic form, are readily derived. For example, from Napier's parts, say α, β, γ, δ, ϵ, we may derive by using the function $\cos^2$ (thus: $x_1=\cos^2\alpha$, $x_2=\cos^2\beta$, ..., $x_5=\cos^2\epsilon$) a set of circular parts of which the rules are: For opposite parts

$$x_1 = 1 - x_3x_4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

for adjacent parts $x_1 + x_3 = 1 + x_1x_2x_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$

This means that the ten equations (five of each type) are such that two of the five x's may be assumed at random, say x_1 and x_2, and then the remaining three x_3, x_4, and x_5 can be so determined as to satisfy all ten of the equations.

Since the second relation is readily derived from the first the question arises whether any equation can be assumed as the relation of adjacent parts. The writer is of the opinion that this is not so, but that some criterion exists, beyond his powers to discover, that would determine this question from a purely algebraic point of view.

6. This led the writer to wonder whether a set of five circular parts existed in which the rules for adjacent parts (and hence, as will be seen, all the rules for the other collocations) are linear algebraic relations.

It may be remarked, as a purely personal matter, that his mode of approach to this question was to assume as the "primary" rule

$$x_1 + bx_2 + x_3 = k$$

(the coefficients of x_1 and x_3 must obviously be the same).

On testing the five equations of this type, in which x_1 and x_2 (for example) are assumed at random, they gave the same values to x_3, x_4, and x_5, provided only that b satisfied the equation

$$b^2 + b = 0 \quad\quad\quad (3).$$

Denoting the roots of this equation by w and w' a set of circular parts exists of which the relation between adjacent parts is

$$x_1 + wx_2 + x_3 = k \quad\quad\quad (4).$$

From this the relation between opposite parts is found to be

$$x_4 - w(x_1 + x_2) = (1 - w)k \quad\quad\quad (5).$$

We have *not* however two cases of sets of circular parts corresponding to the two roots of equation (3), for if we use w' in equation (4), and then transform by taking the order of parts $x_2x_4x_1x_3x_5$ (which converts the collocation "adjacent parts" into "opposite parts"), we have

$$w'x_4 + x_1 + x_2 = k;$$

but, since $ww' = -1$, this is equation (5) with only a different value of k. Thus the two roots of equation (3) correspond to the two "rules", and not to two cases of circular parts.

Geometric interpretation.

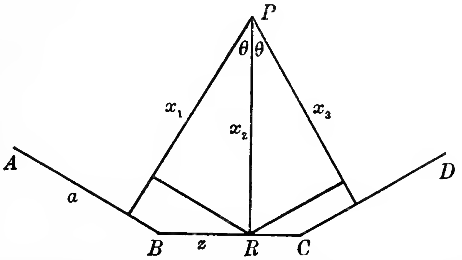
7. The geometric interpretation of this set of circular parts was not far to seek. In fact it consists simply of the

perpendiculars upon the five sides of a pentagon from any point of the plane. As required by the definition of circular parts two of the parts are arbitrary; cyclic interchanges can take place; and the relations between the members of a triad depend only upon the collocation of the three parts.

Relations between the perpendiculars from a point of the plane upon the sides of an n-gon.

8. It is obvious that we may now generalize from 5 to any integer n, and after obtaining the relation between three consecutive perpendiculars from a point of the plane on the sides of an n-gon, obtain by elimination the relation between any triad of the perpendiculars.

Let x_1, x_2, x_3 be the perpendiculars from any point P of its plane upon the three consecutive sides AB, BC, CD of an n-gon. Denote by θ the angle between consecutive perpendiculars, so that $\theta = \frac{2\pi}{n}$; and let a denote the side of the n-gon.



Project x_2 upon x_1 and x_3, and denote by z the distance BR. We have then

$$x_1 = z \sin\theta + x_2 \cos\theta,$$

$$x_3 = (a - z) \sin\theta + x_2 \cos\theta.$$

Adding to eliminate z

$$x_1 + x_3 = a \sin\theta + 2\cos\theta . x_2$$

the relation required.

To simplify the notation, let us now put $a \sin\theta = k$ and $2\cos\theta = c$. Then the primary relation between the circular parts is

$$x_1 + x_3 - c x_2 = k \dots\dots\dots\dots\dots\dots (0,\ 0).$$

This is for the collocation (0, 0) or, when n is put in evidence, (0, 0, $n-3$).

When $n=5$, the comparison with Art. 6 shows that c satisfies the equation $c^2-c=0$. The process in the latter part of Art. 6 is equivalent to using the crossed pentagon and obtains the formula for the collocation (0, 1, 1).

9. In the general case, the "rules" for the other collocations may easily be obtained by elimination from two "rules" already found. For this purpose four parts are selected, two being common to two triads whose collocation rules are known. It follows that the collocations must have a common gap symbol, the parts having this gap being for convenience the middle ones of the four, one of which is to be eliminated. The gap symbol of the result will evidently consist of the remaining gap symbols—one increased by unity.

Thus from two cases of the rule for (0, 0), as applied to the triads $x_1x_2x_3$ and $x_2x_3x_4$ we obtain by eliminating x_3

$$cx_1+(1-c^2)\,x_2+x_4=(1+c)\,k\ldots\ldots\ldots(0,\ 1),$$

which is the rule for the collocation (0, 1).

Again from the triads $x_1x_2x_3$ and $x_2x_3x_5$, cases of the rules already obtained, we obtain

$$(1-c^2)x_1-(2c-c^3)\,x_2{}^2-x_5=-c(1+c)\,k\ldots(0,\ 2),$$

the rule for the collocation (0, 2).

Elimination of x_2 from the same formulæ as those used in finding the rule for (0, 2) gives

$$x_1+(2-c^2)\,x_3+x_5=(2+c)\,k\ldots\ldots\ldots\ldots(1,\ 1),$$

the rule for the collocation (1, 1).

In like manner the writer found the formulæ for the collocations (0, 3) from the triads $x_1x_2x_3$ and $x_2x_3x_6$, eliminating x_3. The result was

$$(2c-c^3)\,x_1+(1-3c^2+c^4)\,x_2-x_6=(c-c^2-c^3)\,k\ldots(0,\ 3).$$

Finally the writer found from the triads $x_1x_3x_4$ and $x_3x_4x_6$, for which the rules had already been found, the result

$$(1-c^2)\,x_1+(1-3c^2+c^4)\,x-cx_6=(1-2c^2-c^3)\,k\ldots(1,\ 2).$$

FACTORIZATION OF N, TREATED AS A BICOMPOSITE, SPECIAL REGARD BEING PAID TO THE SUM OF ITS DIGITS AND TO THE CONSEQUENT POSSIBLE SUMS OF THE DIGITS OF ITS TWIN FACTORS, AFTER CASTING OUT THE NINES.

By *D. Biddle, M.R.C.S.*
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I PROPOSE here to develop the method foreshadowed in the first of my papers published in *The Messenger of Mathematics*, vol. xxviii., pp. 125–131, 148, 149, 192, namely, that which utilizes the sum of the digits of N. This method at once enables us to place N in a definite class, one of six, according to d, the single digit remaining when the nines are cast out. In three of the classes d is odd, and in these $N=18z+d$. In the other three d is even, and in these $N=9z'+d$. In each class there are 3 or 4 cases, differentiated by definite characteristics. In all, there are 21 cases, and if N be a true bicomposite, that is to say, the product of two primes only, its representative, z or z', will appear once only, unless the *case*, as sometimes happens, can be symmetrically divided by one of its diagonals.*

The sixteen commencing representatives for each of the 21 cases will next be exhibited, and, as indicated, the field of each can easily be enlarged, horizontally and vertically, to any required extent, *not that this is needed*.

Class I. $N=18z+1=6n+1$.

Case I. $N=(5+18r)(11+18M)$.

Case II. $N=(7+18r)(13+18M)$.

Case III. $N=(17+18r)(17+18M)$.

Case IV. $N=(1+18r)(1+18M)$.

* It will be observed, however, that more than one N can be represented by the same z or z' in a different class. Thus, we have 33 representing 595 in Class I., Case iii., twice over, owing to symmetry; again in Class II., Case i., as representing $N=299$; again in Class III., Case iii., twice over, owing to symmetry, representing 301; also in Class VI., Case ii., as representing 305.

Case I.

$A_1 = 3,\ B_1 = 11,\ C_1 = 5.$

&c.	&c.	&c.	&c.	&c.
18	83	148	213	&c.
13	60	107	154	&c.
8	37	66	95	&c.
3	14	25	36	&c.

Case II.

$A_2 = 5,\ B_2 = 13,\ C_2 = 7.$

&c.	&c.	&c.	&c.	&c.
26	93	160	227	&c.
19	68	117	166	&c.
12	43	74	105	&c.
5	18	31	44	&c.

Case III.

$A_3 = 16,\ B_3 = 17,\ C_3 = 17.$

&c.	&c.	&c.	&c.	&c.
67	138	209	280	&c.
50	103	156	209	&c.
33	68	103	138	&c.
16	33	50	67	&c.

Case IV.

$A_4 = 20,\ B_4 = 19,\ C_4 = 19.$

&c.	&c.	&c.	&c.	&c.
77	150	223	296	&c.
58	113	168	223	&c.
39	76	113	150	&c.
20	39	58	77	&c.

Class II. $N = 9z' + 2 = 6n - 1.$

Case I. $N = (5 + 18r)(13 + 18M).$

Case II. $N = (7 + 18r)(17 + 18M).$

Case III. $N = (1 + 18r)(11 + 18M).$

Case I.

$A_1 = 7,\ B_1 = 26,\ C_1 = 10.$

&c.	&c.	&c.	&c.	&c.
37	171	305	439	&c.
27	125	223	321	&c.
17	79	141	203	&c.
7	33	59	85	&c.

Case II.

$A_2 = 13,\ B_2 = 34,\ C_2 = 14.$

&c.	&c.	&c.	&c.	&c.
55	197	339	481	&c.
41	147	253	359	&c.
27	97	167	237	&c.
13	47	81	115	&c.

Case III.

$A_3 = 23,\ B_3 = 38,\ C_3 = 22.$

&c.	&c.	&c.	&c.	&c.
89	235	381	527	&c.
67	177	287	397	&c.
45	119	193	267	&c.
23	61	99	137	&c.

Class III. $N=9z'+4=6n+1$.

Case I. $N=(11+18r)(11+18M)$.

Case II. $N=(5+18r)(17+18M)$.

Case III. $N=(7+18r)(7+18M)$.

Case IV. $N=(1+18r)(13+18M)$.

Case I.

$A_1=13,\ B_1=22,\ C_1=22.$

&c.	&c.	&c.	&c.	&c.
79	209	339	469	&c.
57	151	245	339	&c.
35	93	151	209	&c.
13	35	57	79	&c.

Case II.

$A_2=9,\ B_2=34,\ C_2=10.$

&c.	&c.	&c.	&c.	&c.
39	181	323	465	&c.
29	135	241	347	&c.
19	89	159	229	&c.
9	43	77	111	&c.

Case III.

$A_3=5,\ B_3=14,\ C_3=14.$

&c.	&c.	&c.	&c.	&c.
47	169	291	413	&c.
33	119	205	291	&c.
19	69	119	169	&c.
5	19	33	47	&c.

Case IV.

$A_4=27,\ B_4=38,\ C_4=26.$

&c.	&c.	&c.	&c.	&c.
105	251	397	543	&c.
79	189	299	409	&c.
53	127	201	275	&c.
27	65	103	141	&c.

Class IV. $N=18z+5=6n-1$.

Case I. $N=(7+18r)(11+18M)$.

Case II. $N=(17+18r)(13+18M)$.

Case III. $N=(1+18r)(5+18M)$.

Case I.

$A_1=4,\ B_1=11,\ C_1=7.$

&c.	&c.	&c.	&c.	&c.
25	90	155	220	&c.
18	65	112	159	&c.
11	40	69	98	&c.
4	15	26	37	&c.

Case II.

$A_2=12,\ B_2=13,\ C_2=17.$

&c.	&c.	&c.	&c.	&c.
63	130	197	264	&c.
46	95	144	193	&c.
29	60	91	122	&c.
12	25	38	51	&c.

Case III.

$A_3 = 5,\ B_3 = 19,\ C_3 = 5.$

&c.	&c.	&c.	&c.	&c.
20	93	166	239	&c.
15	70	125	180	&c.
10	47	84	121	&c.
5	24	43	62	&c.

Class V. $N = 18z + 7 = 6n + 1.$

Case I. $N = (17 + 18r)(11 + 18M).$

Case II. $N = (13 + 18r)(13 + 18M).$

Case III. $N = (5 + 18r)(5 + 18M).$

Case IV. $N = (1 + 18r)(7 + 18M).$

Case I.

$A_1 = 10,\ B_1 = 11,\ C_1 = 17.$

&c.	&c.	&c.	&c.	&c.
61	126	191	256	&c.
44	91	138	185	&c.
27	56	85	114	&c.
10	21	32	43	&c.

Case II.

$A_2 = 9,\ B_2 = 13,\ C_2 = 13.$

&c.	&c.	&c.	&c.	&c.
48	115	182	249	&c.
35	84	133	182	&c.
22	53	84	115	&c.
9	22	35	48	&c.

Case III.

$A_3 = 1,\ B_3 = 5,\ C_3 = 5.$

&c.	&c.	&c.	&c.	&c.
16	75	134	193	&c.
11	52	93	134	&c.
6	29	52	75	&c.
1	6	11	16	&c.

Case IV.

$A_4 = 7,\ B_4 = 7,\ C_4 = 19$

&c.	&c.	&c.	&c.	&c.
64	125	186	247	&c.
45	88	131	174	&c.
26	51	76	101	&c.
7	14	21	28	&c.

Class VI. $N = 9z' + 8 = 6n - 1.$

Case I. $N = (13 + 18r)(11 + 18M).$

Case II. $N = (7 + 18r)(5 + 18M).$

Case III. $N = (1 + 18r)(17 + 18M).$

Case I.

$A_1 = 15,\ B_1 = 22,\ C_1 = 26.$

&c.	&c.	&c.	&c.	&c.
93	223	353	483	&c.
67	161	255	349	&c.
41	99	157	215	&c.
15	37	59	81	&c.

Case II.

$A_2 = 3,\ B_2 = 10,\ C_2 = 14.$

&c.	&c.	&c.	&c.	&c.
45	163	281	399	&c.
31	113	195	277	&c.
17	63	109	155	&c.
3	13	23	33	&c.

Case III.

$A_3 = 35,\ B_3 = 34,\ C_3 = 38.$

&c.	&c.	&c.	&c.	&c.
149	291	433	575	&c.
111	217	323	429	&c.
73	143	213	283	&c.
35	69	103	137	&c.

In all the cases under each class, the particular column and the particular row in which z or z' appears have each a common difference, and the product of the two differences $= N$ in the case of z, or $4N$ in the case of z'.

Let $A =$ the initial z or z' at the left lower corner of the field in which stands the z or z' derived from N. Let $B =$ the common difference between numbers in the base row. Let $C =$ the common difference between numbers in the extreme left-hand column. Let $r =$ the number of rows from the base (not counting the latter) of that in which the z or z' of N is placed. Let $M =$ the number of columns from the left (not counting the first) in which the z or z' of N is placed.

Then $A + MB =$ the number at the foot of the column in which z or z' stands; and $A + Cr =$ the number at the extreme left of the row in which the same z or z' stands.

The common difference of the said column will be $C + 18M$ for z, but $C + 36M$ for z'.

The common difference of the said row will be $B + 18r$ for z, but $B + 36r$ for z'.

The equations for z are as follows:

$$z = BM + A + (C + 18M)r = Cr + A + (B + 18r)M \text{ ....(1)},$$

$$N = (B + 18r)(C + 18M) = BC + 18BM + 18Cr + 324Mr \text{...(2)},$$

$$= \{[z - (Cr + A)]/M\}\{[z - (BM + A)]/r\} \text{ ....(3)}.$$

The equations for z' are as follows:

$$z' = BM + A + (C + 36M)r = Cr + A + (B + 36r)M \quad\ldots\ldots(4),$$

$$4N = (B + 36r)(C + 36M) = BC + 36BM + 36Cr + 1296Mr \,..(5),$$

$$= \{[z' - (Cr + A)]/M\}\{[z' - (BM + A)]/r\} \quad\ldots\ldots\ldots\ldots\ldots(6).$$

By (2), $z = BM + (BC - d)/18 + (C + 18M)r$ ....(7).

Therefore, by (1), $(BC - d)/18 = A$, for Classes I., IV., V..(8).

By (5), $z' = BM + (BC - 4d)/36 + (C + 36M)r$..............(9).

Therefore, by (4), $(BC-4d)/36 = A$, for Classes II.,III.,VI.(10).

Again, $z - A = (B + 18r)M + Cr$, therefore

$$M = (z - A - Cr)/(B + 18r) \quad\ldots\ldots(11),$$

and $z' - A = (B + 36r)M + Cr$, therefore

$$M = (z' - A - Cr)/(B + 36r) \ldots\ldots(12).$$

Also, by (11), $r = (z - A - BM)/(C + 18M)$ ....(13),

and, by (12), $r = (z' - A - BM)/(C + 36M)$ ....(14).

Marshalling the facts as to N treated by this method, the inception of which took place in the *Messenger*, vol. xxviii., 1898, we have the *class* to which N belongs absolutely given by d, and none of the six classes has more than four distinct *cases*. We begin with a knowledge of z or z' also, and, if the field of these cases were sufficiently extended, we should be able actually to see z or z' in its own exclusive and indisputable position. Moreover, we know A, B, and C for each case. But when the field is not sufficiently extended to present z or z' to our visual organs, M and r must be determined mathematically. These, however, are interdependent, one being discovered the other readily follows. Now, M and r may be equal, except in those "cases" which are symmetrical about the diagonal (N not being a square). But, as a rule, it is advisable to seek first that one of the two which is connected with the larger common difference, whether B or C. It will be found in both numerator and denominator of the value assigned to its fellow, M or r. The process consists in transforming the apparent fraction into an integer, and M and r reveal themselves consentaneously. Thus (11) or (12) should be used when $C > B$, but (13) or (14) when $B > C$.

From the foregoing equations other useful ones can be produced. For instance,

$$Mr = \{z - A - (BM + Cr)\}/18 \quad\ldots\ldots\ldots(15),$$

or

$$Mr = \{z' - A - (BM + Cr)\}/36 \quad\ldots\ldots\ldots(16).$$

From these we can easily find the residues of $BM+Cr$ (mod 18) or (mod 36) respectively.

By (2), (3), we have the following

$$1:M=B+18r:z-A-Cr \quad\text{..........(17)},$$

$$1:r=C+18M:z-A-BM \quad\text{........(18)},$$

where $B+18r=x$ or y, and $C+18M=y$ or x, factors of N.

It is easy, therefore, to see that $M:r$ is approximately that of one factor of N to the other, although BM and Cr are rarely equal. If they were equal we should have $C:B=M:r$, which is not the case (as a rule). The said ratio, however, may help to limit the possible values of M and r. But $18Mr$ is by far the larger constituent of each of the quantities to the right of (17) and (18).

By (1) and (4) we have for different classes

$$18Mr=z-A-BM-Cr \quad\text{...........(19)},$$

and

$$36Mr=z'-A-BM-Cr \quad\text{...........(20)}.$$

In these equations A, B, C are, for each of the 21 "cases", known, besides 18 and z or 36 and z'. Dividing each equation by 18 or 36, there must be a speedy limit to the additions of 18 to the remainder left by $(z-A)/18$, or of 36 to that left by $(z'-A)/36$, before we arrive at $BM+Cr$ in the "case" to which the particular z or z' belongs. Taking a six-figured $N=150809$, $z=8378$, $d=5$, belonging to Class IV., the proper "case" being iii. Here $A=5$, $B=19$, $C=5$, and

$$(z-A)/18=465,$$

with remainder $=3$; also 23 additions of 18 are needed to amount to $19M+5r=417=19.13+5.34$.

Let $Q=$ the quotient, and $t=$ the remainder on division of $z-A$ by 18. Also let $u=$ the number of additions of 18 to t in order to arrive at $BM+Cr$. Then we have

$$z-A=18Q+t \quad\text{.................(21)},$$

$$t+18u=BM+Cr\text{.................(22)},$$

$$Q-u=Mr \quad\text{.......................(23)}.$$

By (22), (23),

$$M=\{t+18u\pm[(t+18u)^2-4BC(Q-u)]^{\frac{1}{2}}\}/2B\text{...(24)}.$$

Also, let $Q'=$ the quotient, and $t'=$ the remainder on division

of $z'-A$ by 36, and let $u'=$ the number of additions of 36 to t in order to arrive at $BM+Cr$. Then we have

$$z'-A=36Q'+t' \quad\text{..................}(25),$$

$$t'+36u'=BM+Cr \quad\text{..................}(26),$$

$$Q'-u'=Mr \quad\text{.......................}(27).$$

By (26), (27),

$$M=\{t'+36u'\pm[(t'+36u')^2-4BC(Q'-u')]^{\frac{1}{2}}\}/2B...(28).$$

On the right of (24) and (28), u and u' are the only unknowns. Moreover, the fact that each quantity under the radical sign is divided into two portions by the minus sign enables us to set a lower limit to the value of u or u'.

Thus, in (24),

$$(t+18u)^2+4BCu-4BCQ=\square\text{.........}(29),$$

and, in (28), $\quad (t'+36u')^2+4BCu'-4BCQ'=\square\text{.........}(30).$

On the left of each the sum of the two first terms must exceed the third, which is entirely known. Applying (29) to the above instance, $N=150809$ shows that $u>22$, and 23 is the correct value.

As an instance of the second kind, let $N=11771$, $N=9.1307+8$ belonging to Class VI., Case II. Here $A=3$, $B=10$, $C=14$, $z'=1307$, $Q'=36$, $t'=8$. And (30) becomes $1296u'^2+1136u'-20096=\square$. This, on reduction, yields $81u'^2+71u'-1256=\square$, which gives $u'=4$, and

$$Mr=Q'-u'=32,\ M=4,\ r=8,\ BM+Cr=152.$$

Hence $\qquad N=149.79=(5+18r)(7+18M),$

and z' is placed in a row, of which the common difference is 298, and in a column whose common difference is 158.

A slight expansion of Case II. of Class VI. exhibits z' of this example in position:

115	413	711	1009	*1307*
101				1149
87				991
73				833
59				675
45	163	281	399	517
31	113	195	277	359
17	63	109	155	201
3	13	23	33	43

It now behoves us to discover which "case" of its class z or z' belongs to, without having befoggedly to test each in turn. We seem to have almost, if not quite, enough facts to go upon, and the process should be brief. Great care, however, must be taken to avoid errors. The rules are few.

(1) Put the values of z, A_1, B_1, C_1 side by side.

(2) Find Q_1 and t_1 on division of $z - A_1$ by 18 (or 36 for z').

(3) Next utilize (29) or (30) for finding the lower limit of u.

(4) Then, by (24) or (28), proceed to discover whether the quantity under the radical sign be a perfect square. One or two trials generally succeed taken above the lower limit found by Rule (3).

(5) If two trials fail, proceed with A_2, B_2, C_2, and so on through all the cases of the particular class.

N.B.—It will be observed that when N is of form $6n+1$, it belongs to a class having 4 cases, but when it is of form $6n-1$ its class has only 3 cases.

If two trials fail in every case, make a third trial with A_1, B_1, C_1, and so on. There is no goal to be reached until the aforesaid perfect square be found.

(6) But when it is found, utilize the fact that its root equals $BM \sim Cr$, whilst the first term under the radical sign in (24) or (28) indicates $(BM + Cr)^2$. The second term under the radical sign, being $4BCMr$, is the difference between the two squares.

Let $N = 6049 = 18.336 + 1$, belonging to Class I., $A_1 = 3$, $B_1 = 11$, $C_1 = 5$, $z - A_1 = 333 = 18.18 + 9$. Therefore $Q_1 = 18$, $t_1 = 9$. The formula (29) determines $u_1 > 2$. Testing 3 it fails in (24), but 4 succeeds, for $t_1 + 18u_1 = 81$, and

$$4BC(Q_1 - u_1) = 220\,(18 - 4),$$

and $81^2 - 220.14 = 59^2$. Therefore

$$B_1M + C_1r = 81, \quad B_1M - C_1r = 59.$$

$$81 - 59 = 2B_1M, \text{ whence } M = 1.$$

$$81 + 59 = 140 = 2C_1r, \quad \text{whence} \quad r = 14.$$

$$x = 5 + 18M = 23\,; \; y = 11 + 18r = 263.$$

But we will not leave the subject without investigating the relations to Case I. of the other cases in the same class. Here therefore I append a conspectus showing for the several

positions (16 in number), exhibited as belonging to the several cases, the additions to, or deductions from, z or z' in Case I. in order to arrive at the similar quantities in the succeeding cases of the same class.

Class I. Relation to Case I.

of Case II.				of Case III.				of Case IV.			
+ 8	10	12	14	+ 49	55	61	67	+ 59	67	75	83
6	8	10	12	37	43	49	55	45	53	61	69
4	6	8	10	25	31	37	43	31	39	47	55
2	4	6	8	13	19	25	31	17	25	33	41

Class II. Relation to Case I.

of Case II.				of Case III.			
+ 18	26	34	42	+ 52	64	76	88
14	22	30	38	40	52	64	76
10	18	26	34	28	40	52	64
6	14	22	30	16	28	40	52

Class III. Relation to Case I.

of Case II.				of Case III.				of Case IV.			
− 40	− 28	− 16	− 4	− 32	− 40	− 48	− 56	+ 26	42	58	74
− 28	− 16	− 4	+ 8	− 24	− 32	− 40	− 48	22	38	54	70
− 16	− 4	+ 8	+ 20	− 16	− 24	− 32	− 40	18	34	50	66
− 4	+ 8	+ 20	+ 32	− 8	− 16	− 24	− 32	14	30	46	62

Class IV. Relation to Case I.

of Case II.				of Case III.			
+ 38	40	42	44	− 5	+ 3	+ 11	+ 19
28	30	32	34	− 3	+ 5	+ 13	+ 21
18	20	22	24	− 1	+ 7	+ 15	+ 23
8	10	12	14	+ 1	+ 9	+ 17	+ 25

Class V. Relation to Case I.

of Case II.				of Case III.				of Case IV.			
− 13	− 11	− 9	− 7	− 45	− 51	− 57	− 63	+ 3	− 1	− 5	− 9
− 9	− 7	− 5	− 3	− 33	− 39	− 45	− 51	+ 1	− 3	− 7	− 11
− 5	− 3	− 1	+ 1	− 21	− 27	− 33	− 39	− 1	− 5	− 9	− 13
− 1	+ 1	+ 3	+ 5	− 9	− 15	− 21	− 27	− 3	− 7	− 11	− 15

Class VI. Relation to Case I.

of Case II.				of Case III.			
−48	−60	−72	−84	+56	68	80	92
−36	−48	−60	−72	44	56	68	80
−24	−36	−48	−60	32	44	56	68
−12	−24	−36	−48	20	32	44	56

The first thing to notice here is, that not only in each row, but in every row of a particular "case" (as related to Case I.) a common difference is exhibited, and also in every column, though not necessarily the difference of the rows.

On examination of the former table of classes and of the factors of N formulated at the head of each, we find that the present common differences equal those between B, C in Case I. and B, C of the particular case under examination.

If only we could arrive at the case, as easily as we arrive at the class, to which N belongs, the present method might almost claim to be a direct method of factorization. And I am sure that the time will come when this will be effected.

In conclusion, however, there is another point to notice, for we have

$$z \text{ or } z' \equiv A + BM (\text{mod } C + 18 \text{ or } 36M) \ldots\ldots (31)$$

$$\equiv A + Cr\,(\text{mod } B + 18 \text{ or } 36r) \ldots\ldots\ldots (32).$$

Considering z alone now, we thus obtain

$$N = 18z + d \equiv 18A + d + 18BM (\text{mod } C + 18M) \ldots (33)$$

$$\equiv BC + 18BM (\text{mod } C + 18M) \ldots\ldots\ldots (34)$$

$$\equiv B(C + 18M)(\text{mod } C + 18M) \ldots\ldots\ldots (35),$$

and, by (32) also, $\equiv 0 \; (\text{mod } C + 18M) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (36)$

$$\equiv 0 \; (\text{mod } B + 18r) \ldots \ldots\ldots\ldots\ldots\ldots\ldots\ldots (37).$$

This proves that z lies on a row whose common difference is one factor of N, and in a column whose common difference is the other factor of N. In the case of z' the common differences are respectively twice the factors of N. But (35) gives us more than this, for it tells us that

$$18(A + BM) + d = B(C + 18M) \ldots\ldots\ldots (38),$$

and $$18(A + Cr) + d = C(B + 18r)\ldots\ldots\ldots\ldots(39),$$

therefore $$M + A/B : r + A/C = C + 18M : B + 18r\ldots.(40),$$

which approximately gives the relation of M and r to the factors of N.

We will now take some notice of the constitution or structure of the said cases.

Those belonging to the z' classes, II., III., VI., consist entirely of odd numbers. But those belonging to the z classes, I., IV., V., have both odd and even numbers in equal proportions. The arrangement of these latter is definite. The diagonal line of numbers, which rises from A, consists entirely of such as are of like character with A (odd or even). Running parallel to this diagonal and on both sides of it are lines of the opposite character, and beyond this other parallel lines similarly alternating in character, so far as the field extends, and there is no limit. The whole series of lines thus described is crossed by another set of parallels of alternating character (odd or even), but not of such uniformly increasing value throughout their length. In fact, after crossing the aforesaid diagonal from below, the value of the numbers in these lines begins to diminish.

Another fact to be taken note of, in regard to z cases, is that we can view them as consisting of intermingled *rhombi*, the four vertices of which are numbers of one character (odd or even) enclosing a central number of the opposite character. The differences of the numbers forming opposite sides of such a rhombus are identical, and this difference may extend to any number of rhombi in succession having common sides.

Examining Class I. as an example of the z classes, and taking first the diagonals of the four cases, we find a definite distinction between them. Rising from A, the first link in the chain $= (B + C + 18)$ always. In the four cases of Class I. the values are 34, 38, 52, 56 respectively. Each successive link is 36 beyond its predecessor. Thus, taking Case I., $34 + 36 = 70$, $70 + 36 = 106$, etc. The "links" are the differences between the successive numbers forming the diagonal. The parallels directly over the aforesaid links are found by adding 18 instead of 36. Consequently we get 34, 52, 70 as differences between numbers in the two first columns, 52, 70, 88 in the next, and so on, the differences being between numbers which are parallel to the diagontl or coincident with it.

If we ascend vertically from $3 \leftarrow 34 \rightarrow 37$ through $8 \leftarrow 52 \rightarrow 60$, and lastly through $63 \leftarrow 250 \rightarrow 313$, we come

to $68 \leftarrow 268 \rightarrow 336$, where 336 is the z for $N = 6049$, which we factorized a short time since. If we ascend in the diagonal, we come to 250, the difference below 268 in the vertical ascent. The absence of 268 from the diagonal is due to the latter proceeding by leaps of 36, whilst the vertical proceeds by leaps of 18. The diagonal contains all the alternate differences of the vertical. These equal differences can be linked together by rhombi as shown above, and all these intervening rhombi would exhibit identical differences between the numbers on their coinciding sides.

Proceeding by the formulæ (31), (32) by deducting the number $(A + BM)$ or $(A + Cr)$ from z, and dividing by the common difference of the particular column or row, we soon find whether z belongs to that case or not*. In regard to $z = 336$, we find, in Case I., that 14 is at the foot of the first column beyond A, and the common difference is 23, which proves successful, being one of the factors of N. We also find $r = 14$, $M = 1$.

It may further be noted that Q, which is the quotient on division of $z - A$ by 18, may be either odd or even. But t, the remainder, depends on the characters of z and A. When both these are odd, or both even, t will be even; but when they differ, t will be odd.

Moreover, when z and A are of similar character, M and r are of similar character also, though not necessarily the same as z and A. But when z and A are of opposite character, so likewise are M and r.

$BM + Cr$ is odd or even with t.

In regard to the classes devoted to z', namely, II., III., VI., although every number in each of the cases is odd, there is considerable similarity as to the relations they bear to each other. It is true they are virtually twice as big in proportion to the size of N as compared with the z numbers. Moreover, the ascent of differences in the diagonal is by leaps of 72, and of the parallels taken vertically (as before) by leaps of 36.

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* The best plan will be to fill in the diagonal, as explained already (in the last paragraph but one) from A until the value of z or z' is just overpassed. We shall then have, for Case i., &c., a member of each of n rows, and also of n columns. The foot values of these, and the left-hand values of the rows, can at once be placed; so that $2n$ trials, carried out according to the instructions, will suffice for any one case, and, when z or z' is found, the factors of N are at once known. It will easily be seen that z or z' cannot lie in any row or column beyond those mentioned. It is further obvious that, in the six, out of twenty-one, cases which are symmetrical about the diagonal, there is no need to examine the n columns as well as the n rows. The possible necessity for $2n$ trials applies, therefore, to only fifteen of the cases. This makes the maximum number of trials required, for N of a given size, about the same in all classes. Moreover, roughly, $n^2 = Mr$, so that n varies much as $\sqrt{N}$ in its particular class.

Again, there are rhombi noticeable in these cases as well as in those of z, having the same differences on opposite sides, but the central z' no longer differing in character. Nevertheless, the rhombi intermingle in a precisely similar manner.

Since z stands in a row and a column, the common differences of which are the two factors of N, and since the foot of the column is $A+BM$, whilst the beginning of the row is $A+Cr$, the following congruences can only be true when A, B, C belong to the same case as z. See (1).

$$(z-A-BM)(z-A-Cr) \equiv 0 \pmod{N} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(41),$$

$$(z-A)^2-(z-A)(BM+Cr)+BC.Mr \equiv 0 \pmod{N} \quad\ldots\ldots(42),$$

$$(18Q+t)^2-(18Q+t)(18u+t)+BC(Q-u) \equiv 0 \pmod{N} \ldots(43),$$

$$(18Q+t)\{18(Q-u)\}+BC(Q-u) \equiv 0 \pmod{N} = N.Mr\ldots(44).$$

By reference to (20), and the remarks below it, also to (25), (26), (27), and again to (4) and (5), we have, for the cases in which z' figures, the following

$$(z'-A-BM)(z'-A-Cr) \equiv 0 \pmod{4N} \quad\ldots\ldots\ldots\ldots\ldots\ldots(45),$$

$$(z'-A)^2-(z'-A)(BM+Cr)+BC.Mr \equiv 0 \pmod{4N}\ldots\ldots(46),$$

$$(36Q'+t')^2-(36Q'+t')(36u'+t')+BC(Q'-u') \equiv 0 \pmod{4N}\ldots(47),$$

$$(36Q'+t')\{36(Q'-u')\}+BC(Q'-u') \equiv 0 \pmod{4N} = 4N.Mr\ldots(48).$$

When N is large considerable aid can be derived from the use of the following formulæ

$$(BM+Cr-t)/18 = u, \text{ with } Q-u = Mr,$$

or
$$(BM+Cr-t')/36 = u', \text{ with } Q'-u' = Mr.$$

Let us try Professor Jevons' 10-figured number in this way, $N = 8616460799$. Here $d=2$, so that N belongs to Class II. It also belongs, as we shall see, to Case i. $z' = 957384533$, $A=7$, $B=26$, $C=10$, $Q'=26594014$, $t'=22$. In this Case i., $B+C=36$. The lower limit of u', namely, $u'-v'=4619$, and we can take $M_1 = r_1 = u'-v'$, as follows

$$(26.4619+10.4619-22)/36 = u'-v'.$$

We can eliminate the small fraction by observing that $10.13-22=3.36$. Consequently we obtain as possible

$$(26.4619+10.4632)/36 = 4622.$$

But
$$4619.4632 < Q'-4622.$$

Now $10.18 = 5.36$ here if we need small additions for adjustment. But first let us add k to both M_1 and $(r_1 + 13)$ and also to $(u' - v' + 3)$ until $Q' - u'$ is nearly reached.

$$(4619 + k)(4632 + k + 18p) = Q' - (4622 + k + 5p),$$

$$k^2 + (9252 + 18p)\,k = 26594014 - (4622 + 5p)$$
$$- (4619.4632) - 18.4619p,$$

$$k + 9\,(514 + p) = (26594060 + 121p + 81p^2)^{\frac{1}{2}}.$$

Taking $p = 19$, $\quad k + 4797 = (26625600)^{\frac{1}{2}} = 5160,$

$$k = 363, \text{ and } 4619 + 363 = 4982 = M,$$

whilst $\qquad 4632 + 363 + 342 = 5337 = r,$

$$N = (18.4982 + 5)(18.5337 + 13) = 89681.96079.$$

A DIFFERENTIAL EQUATION OCCURRING IN THE THEORY OF THE PROPAGATION OF WAVES.

By *Dr. H. Bateman.*

§ 1. The mathematical theory of the propagation of waves in a non-dispersive medium has been discussed in a general manner by E. Vessiot*, whose results may be used in Einstein's theory of gravitation and in other theories in which the form of an elementary wave issuing from a point (x, y, z) at time t is determined initially by an equation of type

$$\sum_{m=0}^{3} \sum_{n=0}^{3} g_{mn}(x_0, x_1, x_2, x_3)\, dx_m dx_n = 0 \text{.........(1)},$$

wherein $x_0 = t$, $x_1 = x$, $x_2 = y$, $x_3 = z$.

In Einstein's theory of gravitation the equations for the propagation of electromagnetic waves may be written in the form

$$\left.\begin{aligned} \operatorname{rot} \mathbf{H} &= \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}, & \operatorname{div} \mathbf{D} &= 0 \\ \operatorname{rot} \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, & \operatorname{div} \mathbf{B} &= 0 \end{aligned}\right\} \text{............(2)},$$

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* *Bull. de la Soc. Mathématique*, t. 34 (1906), p. 230; *Annales de l'École normale* (3), t. 26 (1909), p. 405; *Comptes Rendus*, t. 166 (1918), p. 349.

where the vectors **H**, **D**, **E** and **B** are connected by relations which are linear in their components, and are just sufficient to determine the vectors **E** and **B** uniquely when **H** and **D** are given. These linear relations are similar to the relations connecting the co-ordinates of two lines which are reciprocal polars with respect to the quadric (1) when dx_0, dx_1, dx_2, dx_3 are treated as homogeneous co-ordinates and the quantities g_{mn} as coefficients. The relations are in fact identical with those proposed by the present author* as a scheme which can be combined with (2) so as to give a type of wave propagation which can be described by means of equation (1).

The partial differential equation of the characteristics of the field equations (2) and our scheme of linear relations is

$$\Omega(\theta, \theta) \equiv \sum_{m=0}^{3} \sum_{n=0}^{3} g^{(mn)}(x_0, x_1, x_2, x_3) \frac{\partial\theta}{\partial x_m} \frac{\partial\theta}{\partial x_n} = 0 \quad \ldots (3),$$

where the quantities g_{mn}, $g^{(mn)}$ are connected by the relations

$$\sum_{n=0}^{3} g_{mn} g^{(sn)} = 0 \quad m \neq s,$$

$$= 1 \quad m = s,$$

$$g^{(mn)} = g^{(nm)}.$$

In Einstein's theory the functions $g_{mn}(x_0, x_1, x_2, x_3)$ are not arbitrary, but are solutions of certain gravitational equations. We may still, however, regard equation (3) as the partial differential equation of the characteristics, and so we shall not concern ourselves directly with the gravitational equations. Our object is to study the equation (3) when the coefficients $g^{(mn)}$ are unrestricted, and to determine some cases in which this equation possesses a solution of type

$$\theta = f(\alpha, \beta) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

where f is an arbitrary function of two parameters α and β, which are functions of x_0, x_1, x_2, and x_3.

§ 2. It is easy to see that solutions of type (4) exist when the three partial differential equations

$$\Omega(\alpha, \alpha) = 0, \quad \Omega(\alpha, \beta) = 0, \quad \Omega(\beta, \beta) = 0 \quad \ldots\ldots (5)$$

are compatible. Let us write $\Omega(\alpha, \alpha)$ in the form

$$\sum_{n=0}^{3} a_n \frac{\partial\alpha}{\partial x_n} \sum_{m=0}^{3} b_m \frac{\partial\alpha}{\partial x_m} - \sum_{n=0}^{3} l_n \frac{\partial\alpha}{\partial x_n} \sum_{m=0}^{3} p_m \frac{\partial\alpha}{\partial x_m}$$

* *Proc. London Math. Soc.* (2), t. 8 (1910), p. 223.

then the equations (5) can be satisfied if the four equations

$$\left.\begin{aligned}\sum_{n=0}^{3}(a_n-\lambda l_n)\frac{\partial\alpha}{\partial x_n}=0\\ \sum_{m=0}^{3}(\lambda b_m-p_m)\frac{\partial\alpha}{\partial x_n}=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(6),$$

$$\left.\begin{aligned}\sum_{n=0}^{3}(a_n-\lambda l_n)\frac{\partial\beta}{\partial x_n}=0\\ \sum_{m=0}^{3}(\lambda b_m-p_m)\frac{\partial\beta}{\partial x_m}=0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

can be satisfied, λ being a quantity which is determined by one of these equations. Now these four equations are compatible if the first two equations form a complete system, and the condition for this is that the equation

$$\sum_{m=0}^{3}\sum_{n=0}^{3}\left\{(\lambda b_m-p_m)\left[\frac{\partial a_n}{\partial x_m}-\frac{\partial}{\partial x_m}(\lambda l_n)\right]\frac{\partial\alpha}{\partial x_m}\right.$$
$$\left.-(a_n-\lambda l_n)\left[\frac{\partial}{\partial x_n}(\lambda b_n)-\frac{\partial p_m}{\partial x_n}\right]\frac{\partial\alpha}{\partial x_m}\right\}\ldots\ldots(8)$$

should be a linear combination of the two equations (6). Writing down the conditions which must be satisfied, we obtain two partial differential equations of the first order for λ, and the elimination of λ from these equations will furnish us with the condition which must be satisfied by the coefficients a_n, b_m, l_n, p_m in order that the partial differential equation $\Omega(\theta, \theta)=0$ may possess solutions of the desired type.

§ 3. The analysis in the general case is very complex, and so we shall confine our attention to a simple case which leads to a result of some interest.

Let the equation $\Omega(\theta, \theta)=0$ be

$$\frac{\partial\theta}{\partial x}\frac{\partial\theta}{\partial y}=\nu\frac{\partial\theta}{\partial z}\frac{\partial\theta}{\partial t}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

where ν is some function of x, y, z and t, whose form is to be ascertained. The equations which must form a complete system are now

$$\frac{\partial\alpha}{\partial x}=\lambda\frac{\partial\alpha}{\partial z},\quad \lambda\frac{\partial\alpha}{\partial y}=\nu\frac{\partial\alpha}{\partial t}\ldots\ldots\ldots\ldots\ldots(10),$$

and the two partial differential equations for λ are

$$\frac{\partial\lambda}{\partial t}=\mu\frac{\partial\lambda}{\partial y},\quad \frac{\partial\mu}{\partial x}=\lambda\frac{\partial\mu}{\partial z}\quad\text{...............(11)},$$

where $\mu=\frac{\lambda}{\nu}$. Substituting in the second equation the value of μ given by the first, we obtain an equation which may be written in the form

$$\frac{\partial\lambda}{\partial y}\frac{\partial}{\partial t}\left[\frac{\partial\lambda}{\partial x}-\lambda\frac{\partial\lambda}{\partial z}\right]=\frac{\partial\lambda}{\partial t}\frac{\partial}{\partial y}\left[\frac{\partial\lambda}{\partial x}-\lambda\frac{\partial\lambda}{\partial z}\right]\quad\text{...(12)},$$

and which implies that

$$\frac{\partial\lambda}{\partial x}-\lambda\frac{\partial\lambda}{\partial z}=F(x,\ z,\ \lambda)\quad\text{...............(13)},$$

where F is some function of the three quantities x, z, λ. Treating this as a partial differential equation for λ, we have the Lagrangian equations

$$\frac{dx}{1}=-\frac{dz}{\lambda}=\frac{d\lambda}{F}\quad\text{..................(14)},$$

the solution of which depends upon that of the differential equation

$$\frac{d^2z}{dx^2}+F\left(x,\ z,\ -\frac{dz}{dx}\right)=0\quad\text{............(15)}.$$

Let us suppose that we can find two independent first integrals of this equation, viz.

$$\left.\begin{aligned}c_1&=g_1\left(x,\ z,\ -\frac{dz}{dx}\right)=g_1(x,\ z,\ \lambda)\\ c_2&=g_2\left(x,\ z,\ -\frac{dz}{dx}\right)=g_2(x,\ z,\ \lambda)\end{aligned}\right\}\quad\text{.........(16)},$$

where c_1 and c_2 are arbitrary constants. The general solution of the partial differential equation (13) is then $g_2=\psi(g_1)$, but what we need is really a particular integral involving two arbitrary constants y and t. Such an integral is given by the equation

$$g_2(x,\ z,\ \lambda)=yg_1(x,\ z,\ \lambda)+t\quad\text{...........(17)},$$

which may be regarded as an equation determining λ as a function of x, y, z and t. A possible value of ν is then given by the equation

$$\nu=\lambda\frac{\partial\lambda/\partial y}{\partial\lambda/\partial t}\quad\text{.....................(18)}.$$

Particular values of ν may be obtained by starting with well-known equations $g_2=\psi(g_1)$, which are of Lagrange's type*. Thus if we start with the equation

$$z-2x\frac{dz}{dx}=\psi\left[x\left(\frac{dz}{dx}\right)^2\right],$$

in which $z-2x\dfrac{dz}{dx}=c_1$ and $x\left(\dfrac{dz}{dx}\right)^2=c_2$ are both first integrals of the differential equation

$$2x\frac{d^2z}{dx^2}+\frac{dz}{dx}=0,$$

we obtain the equation

$$x\lambda^2=y\left[z+2x\lambda\right]+t$$

as an equation defining a possible value of λ. This gives

$$\lambda=y\pm\sqrt{\left(\frac{yz+t}{x}+y^2\right)},$$

$$\nu=\lambda\left[z+2xy\pm 2x\sqrt{\left(\frac{yz+t}{x}+y^2\right)}\right]=\lambda\rho \text{ say.}$$

Since $\mu\rho=1$, the equations (10) and (11) indicate that

$$\alpha=G(\rho, y, t).$$

Also it is easy to verify that in the present case

$$\frac{\partial\rho}{\partial y}=\rho\frac{\partial\rho}{\partial t};$$

hence since

$$\frac{\partial\alpha}{\partial y}=\rho\frac{\partial\alpha}{\partial t},$$

we must have

$$\frac{\partial G}{\partial y}=\rho\frac{\partial G}{\partial t},$$

and this indicates that

$$\alpha=\Phi\left[\rho y+t, \rho\right],$$

where Φ is an arbitrary function. This is the solution of the desired type.

It may be inferred from the last example that the solution of the desired type is $\theta=f(\alpha, \beta)$, where $\alpha=g_1(x, z, \lambda)$, $\beta=g_2(x, z, \lambda)$, and λ is defined in terms of x, y, z and t by an equation of type (17). To verify that this is the case we have to prove that α satisfies the differential equations (10) and (11).

* Boole's *Differential Equations*, p. 131.

Since $\frac{\partial\lambda}{\partial t} = \mu\frac{\partial\lambda}{\partial y}$, it is evident that the equation $\frac{\partial\alpha}{\partial t} = \mu\frac{\partial\alpha}{\partial y}$ is satisfied. We also have

$$\frac{\partial\alpha}{\partial x} - \lambda\frac{\partial\alpha}{\partial z} = \frac{\partial g_1}{\partial x} - \lambda\frac{\partial g_1}{\partial z} + \frac{\partial g_1}{\partial\lambda}\left(\frac{\partial\lambda}{\partial x} - \lambda\frac{\partial\lambda}{\partial z}\right)$$

$$= \frac{\partial g_1}{\partial x} - \lambda\frac{\partial g_1}{\partial z} + \frac{\partial g_1}{\partial\lambda}F(x, z, \lambda).$$

Now, since $g_1(x, z, \lambda) = c_1$ is a first integral of (15),

$$\frac{\partial g_1}{\partial x} + \frac{\partial g_1}{\partial z}\frac{\partial z}{dx} + \frac{\partial g_1}{\partial\lambda}\frac{d\lambda}{dx} = 0$$

must be identical with the differential equation (15) when λ is put equal to $-\frac{dz}{\partial x}$, and so we infer that $\frac{\partial\alpha}{\partial x} - \lambda\frac{\partial\alpha}{\partial z} = 0$. It should be noticed that equation (17) gives

$$\rho = \frac{1}{\mu} = \frac{\partial\lambda/\partial y}{\partial\lambda/\partial t} = g_1(x, z, \lambda) = \alpha,$$

$$\rho y + t = g_2(x, z, \lambda) = \beta,$$

$$\nu = \lambda\rho = \lambda g_1(x, z, \lambda).$$

§ 4. The problem of finding solutions of the characteristic equation of type (4) is of some interest because when such solutions exist the partial differential equations (2) possess solutions of type

$$\mathbf{H} = \mathbf{h}f(\alpha, \beta), \quad \mathbf{D} = \mathbf{d}f(\alpha, \beta),$$

$$\mathbf{E} = \mathbf{e}f(\alpha, \beta), \quad \mathbf{B} = \mathbf{b}f(\alpha, \beta),$$

where the vectors $\mathbf{h}$, $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{b}$ are independent of the form of the arbitrary function f. When these expressions are substituted in the partial differential equations, relations are obtained which enable $\mathbf{h}$, $\mathbf{d}$, $\mathbf{e}$ and $\mathbf{b}$ to be expressed in terms of the derivatives of α and β, provided α and β satisfy the partial differential equations (5).

These equations cannot be satisfied simultaneously in all cases; for instance, when the equation (1) is of the type

$$\left(1 - \frac{2m}{r}\right)dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2$$

used in Einstein's theory of the gravitational field of a single centre of force, it seems that functions of type α and β do not

exist when m is a constant greater than zero, while, when m is zero, it is well known that they do exist.

In the latter case it is possible for entities of finite size to be propagated along the (rectilinear) rays without any loss of individuality. The non-existence of the functions α and β when $m > 0$ may mean that there is a fundamental difference between the propagation of light in a gravitational field and propagation in free space.

SUMMATION OF q-HYPERGEOMETRIC SERIES.

By *Rev. F. H. Jackson, M.A., D Sc.*

Introduction.

Among summable cases of q-hypergeometric series usually termed Heinean series, from the fact that Heine (*Kugelfunctionen*, vol. i.) first discussed a q-series $\phi(a, b, c, q, x)$ analogous to the ordinary hypergeometric series with three elements $F(a, b; c, x)$, the following series may be of interest. We call the series summable, when it is capable of expression by a finite number of Gamma-functions. In this paper the fundamental series under discussion is a special case of a q-hypergeometric series with thirteen elements. From this series and equivalent q-product many interesting results in elliptic ϑ functions can be deduced, the nature of which is stated below. It is well known that $F(a, b; c, 1)$, symmetrical in two elements a, b, can be expressed as

$$\frac{\Gamma(c-a-b)\,\Gamma(c)}{\Gamma(c-a)\,\Gamma(c-b)},$$

with an analogous form for q-series in terms of the Basic Gamma-function.* The question arises, are similar summations by Gamma or q-Gamma functions possible for series symmetrical in 3, 4, 5 or higher number of elements? In the appendix to this paper this question is discussed, and the conclusion is reached that such summations are only possible

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* *Proc. R. S.*, vol. lxxvi. A.

for series symmetrical in two elements and for series symmetrical in three elements, but that such theorems do not exist for 4 or 5 or higher number of elements. It is remarkable that for symmetry in three elements the series is a case of hypergeometric series with nine elements. Many special cases of summation of hypergeometric series have been considered by various writers, to which I give reference here.

F. Morley, *Proc. London Math. Soc.*, vol. xxxiv.
A. C. Dixon, *Proc. London Math. Soc.*, vol. xxxv.
Saalschutz, *Zeitschrift für Math.*, vol. xxxv.
Dougall, *Proc. Edin. Math. Soc.*, vol. xxxv.
Watson, *Camb. Phil. Trans.* (1910).
Rogers, *Proc. London Math. Soc*, vol. xxxvi.
Jackson, *Messenger of Math.* (1910), p. 151.
" *Proc. Lond. Math. Soc.*, ser. 2, vol. ii, p. 219.
" *Amer. Journ. Math.*, vol. xxxii., 307.
" *Proc. R.S.*, vol. lxxvi., A.

The principal product and equivalent series discussed in this paper may be expressed as follows

$$\frac{[y+z+c+1]_n\,[x+z+c+1]_n\,[x+y+c+1]_n\,[c+1]_n}{[x+c+1]_n\,[y+c+1]_n\,[z+c+1]_n\,[x+y+z+c+1]_n}$$

$$= 1 + \sum_{r=1}^{n} (-1)^r \frac{[c+2r]}{[c]} \cdot \frac{[c]_r}{[r]!}$$

$$\times \frac{[-x]_r\,[-y]_r\,[-z]_r\,[-n]_r}{[x+c+1]_r\,[y+c+1]_r\,[z+c+1]_r\,[x+y+c+1]_r} \cdot \frac{[x+y+z+2c+n+1]_r}{[-x-y-z-c-n]_r}\, q^n \quad \ldots\ldots(1),$$

in which $[c]_r$ denotes $[c][c+1]\ldots[c+r-1]$, $[c]$ being the basic number $(q^c-1)/(q-1)$.

We note the symmetry of the product in x, y, z and the series in x, y, z, n, but n is of course a positive integer.

This q-hypergeometric series is a very general one, and is a finite case of a q-hypergeometric series with thirteen elements.

A great number of interesting particular cases may be deduced therefrom; for instance, in ϑ function theory

$$A \, . \prod_{1}^{\infty} \frac{(1-2azq^r \cos 2u + a^2z^2q^{2r})}{(1-2aq^r \cos 2u + a^2q^{2r})}$$

$$= 1 + \sum_{n=1}^{\infty} q^n \frac{(1-aq^{2n})}{(1-aq^n)} \cdot \frac{(1-aq^n)!}{(1-q^n)!} \frac{[1-2q^{r-1}\cos 2u + q^{2r-2}]^n}{[1-2aq^r \cos 2u + a^2q^{2r}]_1^n}$$

$$\times \frac{(z-1)(z-q)\ldots(z-q^{n-1})}{(1-azq^n)!}\, a^n \ldots\ldots(2),$$

which is interesting in that the factors in the series, containing z, are separate from the factors containing $\cos 2u$. We note

$$A = \prod_{r=1}^{\infty} (1-aq^r)^2/(1-azq^r)^2.$$

Again, a special case of this is

$$A \cdot \prod_{1}^{\infty} \frac{(1-2cq^r \cos 2u + c^2q^{2r})}{(1-2q^r \cos 2u + q^{2r})}$$

$$= 1 + \Sigma 4q^n(1+q^n)\frac{\sin^2 u}{(1-2q^n \cos 2u + q^{2n})} \cdot \frac{(c-1)(c-q)\dots(c-q^{n-1})}{(1-cq^n)!}c^n \quad \dots\dots(3),$$

A being $\prod_{1}^{\infty} (1-q^r)^2/(1-cq^r)^2$.

Changing q to q^2 and putting $c = q^{-1}$ we have an expression for the quotient of the two functions ϑ_4, ϑ_1, namely,

$$Cq^{\frac{1}{4}} \operatorname{cosec} u \frac{\vartheta_4(u)}{\vartheta_1(u)} = 1 + \sum_{n=1}^{\infty} 4q^{2r}(1+q^{2n})\frac{\sin^2 u}{(1-2q^{2n}\cos 2u + q^{4n})} \quad \dots(4),$$

in which $C = \prod_{1}^{\infty} (1-q^{2r})^2/(1-q^{2r-1})^2$.

Putting $u = \frac{1}{2}\pi$,

$$\prod_{1}^{\infty} \frac{(1+q^{2r-1})^2(1-q^{2r})^2}{(1-q^{2r-1})^2(1+q^{2r})^2} = 1 + \frac{4q}{1+q^2} + \frac{4q^2}{1+q^4} + \dots,$$

which, being easily verified for the first nine or ten terms, affords confirmation of the correctness of the work.

Some time ago Prof. Morley showed that

$$1 + c^3 + \left\{\frac{c(c+1)}{2!}\right\}^3 + \dots = \frac{\Gamma\{1-\frac{1}{2}(3c)\}}{\Gamma(1-c)\Gamma(1+c)\Gamma(1-\frac{1}{2}c)}.$$

The q-function generalization of this is a particular case of the theorem (1), but, to economise space, is not stated here. It will be found in article (7) *infra*.

If we make s infinite, then, subject to conditions for convergence of the product and series, viz. $|q| < 1$, we can deduce

$$\prod_{r=1}^{\infty} \frac{(1-axyq^r)(1-axzq^r)(1-ayzq^r)(1-aq^r)}{(1-azq^r)(1-ayq^r)(1-axq^r)(1-axyzq^r)}$$

$$= 1 + \sum_{r=1}^{\infty} q^r \frac{(1-aq^{2r})}{(1-aq^r)} \cdot \frac{(1-aq^r)!}{(1-q^r)!} X_r Y_r Z_r \cdot a^r \dots(5),$$

where X_r stands for $\dfrac{(x-1)(x-q)\dots(x-q^{r-1})}{(1-axq^r)!}$ and similarly for Y and Z.

On showing this theorem to Prof. L. J. Rogers as one of interest in "combinatory analysis", he was able to identify it as a theorem given by him*, which does not, I think, in his notation show the symmetry so clearly as the above form does. Prof. Rogers points out that there are one or two obvious errata in his paper, which I state here.†

(1)

Throughout the paper the following notation will be used

$[a]$	denotes	$(1-q^a)/(1-q)$,
$[a]_n$	,,	$[a]\,[a+1]\ldots[a+n-1]$,
$[n]\,!$	,,	$[1]\,[2]\ldots[n]$,
$[1-aq^n]\,!$	,,	$(1-aq)(1-aq^2)\ldots(1-aq^n)$,
$(x-aq^n)\,!$	,,	$(x-aq)(x-aq^2)\ldots(x-aq^n)$.

The following simple lemma will be required in the course of the proof of theorem (1). The q factorial product

$$[x]\,[x+1]\ldots[x+n-1]$$

may be expressed as

$$[x]^n+a_{n-1}\,[x]^{n-1}+\ldots+a_2\,[x]^2+a_1\,[x]+a_0,$$

that is, a rational integral function of degree n in $[x]$, the coefficients a_0, a_1, a_2, ... not involving x. Thus

$$[x]\,[x+1]\equiv q\,[x]^2+[x],$$

whence

$$[x][x+1][x+2]=q^3\,[x]^3+\frac{2q^3-q^2-q}{q-1}\,[x]^2+[2]\,[x],$$

and, by induction, we can proceed from the truth assumed for

$$[x]\,[x+1]\ldots[x+n-1]$$

to the truth for

$$[x]\,[x+1]\ldots[x+n].$$

Of course, q^x in $[x]$ has only its principal value attached to it.

We can now assert that any two rational integral factorial functions of degree n in $[x]$, which are equal for n particular values of x, and, moreover, have equal coefficients of $[x]^n$, are identities.

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* *Proc. L.M.S.*, vol. xxvi. (1894).

† *Proc. L.M.S.*, vol. xxvi., p. 29, second equation from the top of the page, the denominator factors in u and v should be $1-\lambda uq^{\frac{1}{2}}$, $1-\lambda vq^{\frac{1}{2}}$, $1-\lambda uq^{\frac{3}{2}}$, $1-\lambda vq^{\frac{3}{2}}$, &c., instead of having u's and v's only.

It may seem a fault of form in the following proof of the theorem numbered (1), in the introduction, that the proof involves very considerable *à priori* knowledge of the form of the series. In the appendix I indicate how such forms are very simple to arrive at. I place this work in the appendix so that the main argument may not be interfered with. In that place too, we arrive at an interesting negative result, viz. that there are not extensions of the theorem for the case of four or higher number of variables with symmetry among themselves.

(2)

Summation of a finite case of q-hypergeometric series with thirteen elements.

Consider $$[x+a]\,[x+b].$$

If we write $-x-a-b$ for x, the product becomes

$$[x+a]\,[x+b]\,q^{-2x-a-b},$$

so that further

$$[x+a+1]\,[x+a+2]\ldots[x+a+s]\times[x+b+1]\ldots[x+b+s]$$

becomes (on substituting $-x-a-b-s-1$ for x)

$$[x+a+1]_s\,[x+b+1]_s\,q^{-2sx-2sa-2sb-s^2-s},$$

and the factorial function

$$\frac{[x+a+1]_n\,[x+b+1]_n\,[c+1]_n\,[d+1]_n}{[x+c+1]_n\,[x+d+1]_n\,[a+1]_n\,[b+1]_n}\quad\ldots.(6),$$

which has symmetry in x and n when x is a positive integer, has also the property of remaining invariant when $-x-a-b-n-1$ is substituted for x, provided $a+b=c+d$.

Now consider a series

$$1+A_1\frac{[x]\,[n]\,[x+a+b+n+1]}{[x+c+1]\,[n+c+1]\,[x+d+n]}$$
$$+A_2\frac{[x]\,[x-1]\,[n]\,[n-1]\,[x+a+b+n+1]\,[x+a+b+n+2]}{[x+c+1]\,[x+c+2]\,[n+c+1]\,[n+c+2]\,[x+d+n]\,[x+d+n-1]}$$
$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

in which A_1, A_2, ... are independent of x and n. This series is invariant for the substitution $-x-a-b-n-1$ for x when $a+b=c+d$. Let us assume this form as a possible expansion of the product

$$\frac{[x+a+1]_n\,[x+b+1]_n\,[c+1]_n\,[d+1]_n}{[x+c+1]_n\,[x+d+1]_n\,[a+1]_n\,[b+1]_n}.$$

Give n the value r, multiply both the product and the series by $[x+c+1]\ldots[x+c+r].[x+d+r]\ldots[x+d+1]$, then, putting $x=-d-1$, we find, owing to the vanishing of all terms of the series except the last, that

$$A_2\frac{[-d-1]\ldots[-d-r].[r]!\,[a+b-d-r]\ldots[a+b-d+2r-1]}{[c+r+1]\,[c+r+2]\ldots[c+2r]}$$
$$=\frac{[a-d]_r\,[b-d]_r\,[c+1]_r\,[d+1]_r}{[a+1]_r\,[b+1]_r},$$

whence, remembering that $a+b=c+d$, we have, after a little reduction,

$$A_r=(-)^r\frac{[c+2r]}{[c]}\cdot\frac{[c]_r}{[r]!}\cdot\frac{[a-d]_r\,[b-d]_r}{[a+1]_r\,[b+1]_r}\,q^{\frac{1}{2}\{2rd+r(r+1)\}}.$$

Having in this way determined a suitable form for the coefficients A_0, A_1, ... we proceed by induction.

The identity of the product and series is easily verified for $n=0$, $n=1$. Let us assume the truth of the identity when $n=0, 1, 2, \ldots, (r-1)$, namely, for r particular values of n; then, owing to symmetry in a and n (n a positive integer), this assumption also covers the truth of the identity for r particular values of a, namely, for $a=0, 1, 2, \ldots, (r-1)$.

The functions are invariant for the substitution

$$-x-a-b-n-1$$

for x, so that our assumption also covers the case of the truth of the identity for other r values, namely,

$$a=x+b+r+1,\ x+b+r+2,\ \ldots,\ x+b+2r.$$

If we multiply both product and series by

$$[x+c+1]\ldots[x+c+r]\times[x+d+1]\ldots[x+d+r],$$

on substituting $-d-1$ for x, all terms vanish except the last, and we obtain from the product the following expression

$$\frac{[a-d]_r\,[b-d]_r\,[c+1]_r\,[d+1]_r}{[a+1]_r\,[b+1]_r},$$

and from the series

$$\frac{[-d-1]\ldots[-d-r].[c+r]\ldots[c+2r-1]}{[c+r+1]\,[c+r+2]\ldots[c+2r]}$$
$$\times(-1)^r\,[r]!\,\frac{[c+2r]}{[c]}\cdot\frac{[c]_r}{[r]!}\cdot\frac{[a-d]_r\,[b-d]_r}{[a+1]_r\,[b+1]_r}\,q^{\frac{1}{2}\{2rd+r(r+1)\}}.$$

A little reduction shows that these are identical expressions, so that we have two rational integral functions of degree $2r$ with respect to $[a]$, which are equal for $2r$ particular values of a, also having equal coefficients of $[a]^{2r}$. We assert therefore the following identity for all integral positive values of n, provided $a+b=c+d$,

$$\frac{[x+a+1]_n\,[x+b+1]_n\,[c+1]_n\,[d+1]_n}{[x+c+1]_n\,[x+d+1]_n\,[a+1]_n\,[b+1]_n}$$

$$\equiv 1+\sum_{r=1}^{r=n}(-1)^r\frac{[c+2r]}{[c]}\cdot\frac{[c]_r}{[r]!}\cdot\frac{[a-d]_r\,[b-d]_r}{[a+1]_r\,[b+1]_r}\cdot\frac{[x]\ldots[x-r+1]}{[x+c+1]_r}$$

$$\times\frac{[n]\ldots[n-r+1]}{[n+c+1]_r}\cdot\frac{[x+a+b+n+1]_r}{[x+d+n]\ldots[x+d+n-r+1]}\,q^{\frac{1}{2}\{rd+r(r+1)\}},$$

which is a particular case of a, a hypergeometric series with thirteen elements, and is the fundamental theorem in the present paper numbered (1) in the introduction.

(3)

The theorem can be thrown, however, into a more symmetrical form, which we will proceed to do before considering the special cases of the series.

Writing $y+c$ for a,

$z+c$ for b,

$y+z+c$ for d,

we have

$$\frac{[y+z+c+1]_n\,[x+y+c+1]_n\,[x+y+c+1]_n\,[c+1]_n}{[x+c+1]_n\,[y+c+1]_n\,[z+c+1]_n\,[x+y+z+c+1]_n}$$

$$\equiv 1+\sum_{r=1}^{r=n}(-)^r\frac{[c+2r]}{[c]}\cdot\frac{[c]_r}{[r]!}\cdot\frac{[-x]_r\,[-y]_r\,[-z]_r\,[-n]_r}{[x+c+1]_r\,[y+c+1]_r\,[z+c+1]_r\,[n+c+1]_r}$$

$$\cdot\times\frac{[x+y+z+2c+n+1]_r}{[-x-y-z-c-n]_r}\,q^r\ldots.(8).$$

We note the symmetry of the series in x, y, z, n, but n is restricted to positive integral values.

(4)

The series and product, when we make the integer n great beyond limit are both convergent for $|q|<1$, and in this case

we may write a theorem

$$\prod_{r=1}^{\infty} \frac{(1-axyq^r)(1-axzq^r)(1-ayzq^r)(1-aq^r)}{(1-azq^r)(1-ayq^r)(1-axq^r)(1-axyzq^r)}$$

$$=1+\sum_{r=1}^{\infty} q^r \frac{(1-aq^{2r})}{(1-aq^r)} \cdot \frac{(1-aq^r)!}{(1-q^r)!} X_r Y_r Z_r a^r \ldots (9),$$

where X_r stands for

$$\frac{(x-1)(x-q)\ldots(x-q^{r-1})}{(1-axq^r)!}$$

and similar expressions for y and z are Y_r, Z_r.

This theorem is obtained from (8) by making n infinite and replacing q^x by x, q^y by y, q^z by z, q^c by a, which theorem was given by Prof. L. J. Rogers in another notation, as referred to in the introduction to this paper.

(5)

There are many results of interest contained in the forms (8) and (9), which are, I hope, new, so it may be of interest to give them here.

A theorem in ϑ functions.

If we put

$$x = e^{2iu},$$
$$y = e^{-2iu},$$

we obtain

$$A \prod_{1}^{\infty} \frac{(1-2azq^r\cos 2u + a^2z^2q^{2r})}{(1-2aq^r\cos 2u + a^2q^{2r}}$$

$$=1+\sum_{n=1}^{\infty} q^n \frac{(1-aq^{2n})}{(1-aq^n)} \cdot \frac{(1-aq^n)!}{(1-q^n)!} \cdot \frac{[1-2q^{r-1}\cos 2u+q^{2r-2}]_1^n}{[1-2aq^r\cos 2u + q^{2r}a^2]_1^n}$$
$$\times \frac{(z-1)(z-q)\ldots(z-q^{n-1})}{(1-azq^n)!} a^n \ldots (10),$$

which is an interesting form in that the factors, in the series, containing z, are separate from the factors which contain $\cos 2u$.

A denotes $\Pi(1-aq^r)^2/(1-azq^r)^2$.

On putting $a=1$, $z=c$, we obtain

$$\prod_{1}^{\infty} \frac{(1-2cq^r\cos 2u+c^2q^{2r})}{(1-2q^r\cos 2u+q^{2r})} \cdot \frac{(1-q^r)^2}{(1-cq^r)^2}$$

$$=1+\sum_{n=1}^{\infty} 4q^n \frac{\sin^2 u}{(1+q^n)(1-2q^n\cos 2u+q^{2n})} \cdot \frac{(c-1)(c-q)\ldots(c-q^{n-1})}{(1-cq^n)!}$$
$$\ldots\ldots(11).$$

(6)

On changing q to q^2 and putting $c=q^{-\frac{1}{2}}$, we have an expression for the quotient of two functions theta, namely,

$$Aq^{\frac{1}{4}}\operatorname{cosec} u\,\frac{\vartheta_4(u)}{\vartheta_1(u)}=1+\sum_{n=1}^{\infty}4q^n(1+q^{2n})\frac{\sin^2 u}{(1-2q^{2n}\cos 2u+q^{4n})} \qquad \ldots\ldots(12),$$

in which $$A=\prod_1^{\infty}(1-q^{2r})^2/(1-q^{2r-1})^2,$$

and, on putting $u=\frac{1}{2}\pi$, we have

$$\prod_1^{\infty}\frac{(1+q^{2r-1})^2(1-q^{2r})^2}{(1-q^{2r-1})^2(1+q^{2r})^2}=1+\frac{4q}{1+q^2}+\frac{4q^2}{1+q^4}+\frac{4q^3}{1+q^6}+\ldots.$$

I have verified this up to q^9, and this serves as verification of the preceding work. Up to q^{36} the series is

$$1+4q+4q^2+4q^4+8q^5+4q^8+4q^9+8q^{10}+8q^{13}+4q^{16}+8q^{17}+4q^{18}$$
$$+8q^{20}+8q^{25}+8q^{26}+8q^{29}-4q^{30}+4q^{32}+8q^{34}+8q^{36}+\ldots.$$

The series on the right is also an expansion of ϑ_3^2.

(7)

The following form of the theorem is of interest in the notation of my basic* Gamma-function

$$\frac{\Gamma_q(y+c+1)\Gamma_q(x+c+1)\Gamma_q(z+c+1)\Gamma_q(x+y+z+c+1)}{\Gamma_q(x+z+c+1)\Gamma_q(y+z+c+1)\Gamma_q(x+y+c+1)\Gamma_q(c+1)}$$
$$=1+\sum_{r=1}^{\infty}(-1)^r\frac{[c+2r]}{[c]}\cdot\frac{[c]_r}{[r]!}\cdot\frac{[-x]_r}{[x+c+1]_r}\cdot\frac{[-y]_r}{[y+c+1]_r}\cdot\frac{[-z]_r}{[z+c+1]_r}q^r \qquad \ldots\ldots(13),$$

and from this, putting $z=-\frac{1}{2}c$, $x=y=-c$, we obtain

$$1+\Sigma q^r.\left\{\frac{[c][c+1]\ldots[c+r-1]}{[r]!}\right\}^3.\frac{q^{\frac{1}{2}c+r}+1}{q^{\frac{1}{2}c}+1}$$
$$=\frac{\Gamma_q\{1-\frac{1}{2}(3c)\}}{\Gamma_q(1-c)\Gamma_q(1+c)\Gamma_q(1-\frac{1}{2}c)} \ldots(14).$$

In case $q=1$ this reduces to the sum of cubes of coefficients of x in $(1-x)^{-n}$ given by Professor Morley†

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* *Proc. R.S.*, vol. lxxvi. A, p. 130; also vol. lxxiv., p. 61.
† *Proc. L.M.S.*, vol. xxxiv.

Another interesting series is obtained by putting

$$x = y = z = -c,$$

when we obtain

$$1 + \sum_{r=1}^{\infty} \frac{[c+2r]}{[c]} \left\{ \frac{[c][c+1]\dots[c+r-1]}{[r]!} \right\}^4 q^r$$

$$= \frac{\Gamma_q(1-2c)}{\Gamma_q(1+c)\,\Gamma_q(1-c)\,\Gamma_q(1-c)\,\Gamma_q(1-c)} \quad \dots\dots(15).$$

Finally, putting $x = y = -c$ and making z infinite, we have

$$\frac{1}{\Gamma_q(1-c)\,\Gamma_q(1+c)}$$

$$= 1 + \sum_{r=1}^{\infty} (-)^n \frac{[c+2n]}{[c]} \left\{ \frac{[c][c+1]\dots[c+n-1]}{[n]!} \right\}^3 q^{n^2-nc} \dots(16).$$

Now I have shown in other papers that

$$\frac{[c]}{\Gamma_q(1-c)\,\Gamma_q(1+c)}$$

is an elliptic Sigma-function closely related to Weierstrass's Sigma-function. Denoting this function $S_q'(\omega c)$ we have

$$\frac{S_q(\omega c)}{[\omega c]} = 1 + \Sigma(-)^n \frac{[c+2n]}{[c]} \cdot \left\{ \frac{[c][c+1]\dots[c+n-1]}{[n]!} \right\}^3 q^{n^2-nc},$$

where

$$\omega^{\frac{1}{2}} = \Gamma_q(\tfrac{1}{2}).$$

This function is such that

$$S_q(c+\omega) = -q^{-c} S_q(c),$$

and has a multiplication theorem*

$$S_q\left(c, \frac{\omega}{n}\right) = \text{const.}\, S_q(c, \omega)\, S_q\left(c + \frac{\omega}{n}, \omega\right) \dots S_q\left(c + \frac{n-1}{n}\omega, \omega\right).$$

We now proceed to show how the form of the series (1), which seems so complex, is yet quite simple to arrive at, and incidentally why such theorems exist only for symmetry in two elements and three elements only.

Appendix.

The form of the q-function products and equivalent series may be arrived at in the following manner. Since

$$\frac{(1-ax)(1-ay)}{(1-a)(1-axy)} = 1 + \frac{(1-x)(1-y)}{(1-a)(1-axy)}\, a,$$

* *Proc. R.S.*, vol. lxxvi. A, p. 131.

we have

$$\frac{(1-ax)^n(1-ay)^n}{(1-a^n)(1-axy)^n}=1+\Sigma\frac{n!}{r!\,n-r!}\frac{(1-x^n)(1-y)^n}{(1-a^n)(1-axy)^n}a^n \text{ (A).}$$

The q factorial product and series (if such exist) must be such as will reduce to this factor by factor when q is made equal to unity.

Now the q factorial product and series corresponding to this are well known, being

$$\prod_1^n\frac{(1-aq^rx)(1-aq^ry)}{(1-aq^r)(1-axyq^r)}=1+\sum_{r=1}^{r=n}\frac{(1-x)_r(1-y)_r(1-q^n)_r}{(1-aq^r)!(1-axyq^r)_r(1-q^r)!}a^rq^{r^2}..\text{(B)},$$

where $(1-x)_r$ denotes $(1-x)(1-xq^{-1})...(1-xq^{-r+1})$. From this we can deduce the q analogue of

$$\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}=1+\frac{\alpha\beta}{1.\gamma}+...,$$

and the formula (A) is of course a particular case of (B), namely, when $q=1$, (B) reducing to (A) term by term, factor by factor, when $q=1$.

Similarly for symmetry in three variables

$$\frac{(1-axy)(1-ayz)(1-axz)(1-a)}{(1-ax)(1-ay)(1-az)(1-axyz)}=1-\frac{(1-x)(1-y)(1-z)(1-a^2xyz)}{(1-ax)(1-ay)(1-az)(1-axyz)}a,$$

and, raising both sides to the n^{th} power, we have

$$\frac{(1-axy)^n(1-ayz)^n(1-axy)^n(1-a)^n}{(1-ax)^n(1-ay)^n(1-az)^n(1-axyz)^n}$$

$$=1+\sum_{r=1}^{n}(-)^r\frac{n!}{r!\,n-r!}\frac{(1-x)^r(1-y^r)(1-z)^r(1-a^2xyz)^r}{(1-ax)^r(1-ay)^r(1-az)^r(1-axyz)^r}a^r \text{ (C),}$$

and a q-hypergeometric product and series symmetrical in x, y, z, if such exist, must have this as special case when $q=1$. The principal theorem of the foregoing paper is the q-theorem, of which (C) is the special case, and may be expressed

$$\prod_{r=1}^{r=n}\frac{(1-axyq^r)(1-axzq^r)(1-azyq^r)(1-aq^r)}{(1-axq^r)(1-ayq^r)(1-azq^r)(1-axyzq^r)}$$

$$=1+\sum_{r=1}^{n}(-)^r\frac{(1-aq^{2r})}{(1-aq^r)}\frac{(1-aq^r)}{(1-q^r)!}\frac{(q^{-n}-1)_r}{(1-axq^n)_r}$$

$$\times\frac{(x^{-1}-1)_r(y^{-1}-1)_r(z^{-1}-1)_r}{(1-axq^r)!(1-ayq^r)!(1-azq^r)!}\frac{(1-axyzq^{n+r})_r}{(x^{-1}y^{-1}z^{-1}q^{-n}-1)_r}q^r.\text{(D),}$$

where $(x^{-1}-1)_r$ denotes $(x^{-1}-1)(x^{-1}-q)\ldots(x^{-1}q^{r-1}-1)$. In case $q=1$ we see this reduces term by term, factor by factor, identically to (C) above.

From the form we can gain almost complete knowledge of the requisite form of expansion of the product (1) in the preceding paper. Supposing such expansion to be possible. The question of course arises, how far can we carry the method in obtaining similar products and equivalent series for four or five or higher number of variables possessing symmetry in the variables? The answer is, that no such forms exist for more than symmetry in three variables. The question is settled conclusively by an examination in the case of four variables of the following product, symmetrical in (x, y, z, ω),

$$\frac{(1-a)(1-axy)(1-axz)(1-ayz)(1-ax\omega)(1-ay\omega)(1-az\omega)(1-axyz\omega)}{(1-ax)(1-ay)(1-az)(1-a\omega)(1-axyz)(1-axy\omega)(1-ayz\omega)(1-axz\omega)}$$

while this can be expressed as

$$1+\frac{(1-x)(1-y)(1-z)(1-\omega)}{(1-ax)(1-ay)\ldots(1-axz\omega)}f(a, x, y, z, \omega),$$

it is not possible to express $f(a, x, y, z, \omega)$ in four factors, viz. of the form $(1-a'xyz)(1-a'yz\omega)(1-a'xz\omega)(1-a'xy\omega)$, which would be necessary if theorems analogous to the preceding are to exist. Of course this does not exclude the possibility of special cases of summation, where particular specified relations exist among the variables x, y, z, ω.

The conclusion we reach is that the theorem

$$\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}=F(\alpha, \beta, \gamma),$$

symmetrical in α, β, has its analogue for symmetry in three elements, and such analogue is a case of hypergeometric series with nine elements, but that theorems with symmetry in four or higher number of elements do not exist.

The binomial expansions (A) and (C) are the simplest cases $(q=1)$ of the q-expansions, if such exist, and only when such binomial expansions with symmetry in $x, y, z, \omega, \ldots$ exist can the q series, from which they may be derived, exist.

ON THE GENERATING FUNCTION OF THE SERIES $\Sigma F(n)q^n$, WHERE $F(n)$ IS THE NUMBER OF UNEVEN CLASSES OF BINARY QUADRATICS OF DETERMINANT $-n$.

By *L. J. Mordell*, Birkbeck College, London.

§ 1. Let $F(n)$ be the number of uneven classes of binary quadratics of determinant $-n$ with the convention that the class $(k, 0, k)$ is reckoned as $\frac{1}{2}$ instead of 1, and that $F(0)$ is zero. It is well known that $F(n)$ is a rather complicated arithmetical function of n, and that one expression for $F(n)$, is given by the formula (if $n > 0$)

$$F(n) = f(d_1) + f(d_2) + \ldots = \Sigma f(d) \ldots\ldots\ldots\ldots (1),$$

where d is any divisor of n such that n/d is equal to the square of an odd integer. Also $f(d)$, the number of properly primitive classes of determinant $-d$, with the same convention as above for the form $(1, 0, 1)$ is given by the infinite series

$$\frac{\pi}{2} f(d) = \sqrt{d} \left\{ \frac{1}{1}\left(\frac{-d}{1}\right) + \frac{1}{3}\left(\frac{-d}{3}\right) + \frac{1}{5}\left(\frac{-d}{5}\right) + \ldots \right\}$$

$$= \sqrt{d} \sum_r \frac{1}{r}\left(\frac{-d}{r}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where r takes all odd positive values and $\left(\frac{-d}{r}\right)$ is the Legendre-Jacobi symbol of quadratic reciprocity.

The infinite series can be expressed in a finite form, but from neither of the formulæ is it obvious why the function $F(n)$ should satisfy a number of simple recurrence formulæ, one of which for example may be stated as

$$F(n) + 2F(n-1^2) + 2F(n-2^2) + \ldots = -\Sigma a + \Sigma b \ldots (3),$$

where the summation on the left-hand side is continued so long as the argument of $F(n-r^2)$ is positive; a refers to any divisor of n which $\leq \sqrt{n}$ and of the same parity as its conjugate divisor; but when $a = \sqrt{n}$, we take $\frac{1}{2}a$ in the summation instead of a; also b refers to any divisor of n whose conjugate divisor is odd.

It seems to me that the investigation of class relation formulæ*, of which equation (3) is only a very special case, would be facilitated by the study of the function

$$\chi(\omega) = \sum_{1}^{\infty} F(n)\, q^n \text{..................... (4)},$$

where $q = e^{\pi i \omega}$, and applying if possible the theory of the modular functions. For this purpose it is necessary to find a relation between $\chi(-1/\omega)$ and $\chi(\omega)$, and with this object in view, I have converted $\chi(\omega)$ into the doubly infinite series given by

$$2\pi\chi(\omega) = \sum_{a,b} \frac{(-1)^{\frac{1}{2}(b-1)} \sum\limits_{s=0}^{b-1} e^{-\pi i a s^2/b}}{\sqrt{b}\,[\sqrt{\{-i(a+b\omega)\}}]^3} \text{.......... (5)},$$

where the radical is taken with a positive real part†, a takes all even values, positive, negative, or zero, and b takes all odd positive values. The series is not absolutely convergent, and the summation is carried out first for a; and then for b in the order $b = 1, 3, 5, \ldots$. If the series had been absolutely convergent, the required relation between $\chi(\omega)$ and $\chi(-1/\omega)$ could have been found at once by summing it first for b and then for a. The conditional convergence, however, makes it rather difficult to find the relation between the two sums. In this connection it may be noted that Eisenstein‡ has found the relation between the sums of semi-convergent series of the type§

$$\Sigma\,(a\omega_1 + b\omega_2)^{-r}, \quad r = 1, 2,$$

when the summation for a and b is carried out in various ways. He has also found the results by means of a definite integral. But from other considerations I have already found that‖

$$\int_{-\infty}^{\infty} \frac{t e^{\pi i \omega t^2} dt}{e^{2\pi t} - 1} = -2\chi(\omega) + \frac{2\sqrt{(-i\omega)}}{\omega^2}\,\chi(-1/\omega) + \tfrac{1}{4}\theta_{00}^{\,3}(0, \omega),$$

* See my paper "On Class Relation Formulæ", *Messenger of Mathematics*, vol. xlvi. (1916), for references and some account of the subject.

† This convention applies throughout this paper, and here as elsewhere refers to the quantity inside the square bracket as well as to $\sqrt{b}$.

‡ "Genaue Untersuchung der unendlichen Doppelproducte", *Crelle*, vol. xxxv. (1847).

§ See also Hurwitz, "Grundlagen einer independenten Theorie der elliptischen Modulfunctionen, etc.", *Mathematische Annalen*, vol. xviii. (1881).

‖ "On some series whose nth term involves the numbers of classes of binary quadratics of determinant $-n$", *Messenger of Mathematics*, vol. xlix., 1919.

where

$$\theta_{00}(0, \omega) = 1 + 2q + 2q^4 + 2q^9 + \ldots,$$

and it seems to me that Eisenstein's method applied to equation (5) would only lead to this result. Nevertheless the series (5) is of some interest not only from the fact that it reveals the nature of the singularities of the function $\chi(\omega)$, but also because the proof of it shows the method of summing the series which give the number of representations of a number as the sum of an odd number of squares*.

If it were legitimate to sum the series (5) by grouping together the terms in which a and b have a common factor, on noting that

$$1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} \ldots = \frac{\pi}{4},$$

$$\sum_{s=0}^{b-1} e^{-\pi i a s^2/b} = i^{\frac{1}{2}(b-1)} \sqrt{b} \left(\frac{a}{b}\right),$$

when a is even and prime to b, and where $\left(\frac{a}{b}\right)$ is the Legendre-Jacobi symbol of quadratic reciprocity, we should find formally that

$$8\chi(\omega) = \Sigma\Sigma \frac{i^{-\frac{1}{2}(b-1)} \left(\frac{a}{b}\right)}{[\sqrt{\{-i(a + b\omega)\}}]^3} \ldots\ldots\ldots\ldots(5a),$$

where a and b are as before, but, while I think that now also the summation is carried out first for a, I cannot prove it.† If a is not prime to b, $\left(\frac{a}{b}\right) = 0$, but in the special case when $a = 0$, $b = 1$, we must replace $\left(\frac{a}{b}\right)$ by 1.

§ 2. The doubly infinite series (5) for $\chi(\omega)$ was found‡ originally by substituting for $F(n)$ the infinite series given by equations (1) and (2), and involved some rather intricate reduction.

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* See my paper, "On the representations of a number as the sum of an odd number of squares", *Transactions of the Cambridge Philosophical Society*, vol. xxii. (1919), pp. 367, 368, referred to hereafter as *T.C.P.S.*

† I have since discovered a proof which I add in § 6.

‡ As noted in my paper, *T.C.P.S.*, p. 369.

But when I arrived at the series (5) I noticed that it was of the type considered by Prof. Hardy* and myself† in connection with the number of representations of a number as a sum of an odd number of squares, and which had been converted into a power series in q of a type rather different in form from that in equation (4). Thus we have

$$\chi(\omega) = \sum_{M=1}^{\infty} A_M \sqrt{M} q^M \text{ ....(6)},$$

where

$$A_M = \frac{f(1)}{1} - \frac{f(3)}{3} + \frac{f(5)}{5} \dots = \sum_{b \text{ odd}} (-1)^{\frac{1}{2}(b-1)} f(b)\, b^{-1},$$

and $f(b)$ denotes the number of solutions of the congruence

$$s^2 \equiv M \pmod{b}.$$

We shall show presently that

$$A_M \sqrt{M} = F(M),$$

but it is first necessary to justify the methods leading to equation (6) owing to the conditional convergence of the series involved therein. The series‡ (6) was found by summing (5) with respect to a. This gave

$$\chi(\omega) = \Sigma\Sigma (-1)^{\frac{1}{2}(b-1)} q^{\theta + b\sigma} (\theta + b\sigma)^{\frac{1}{2}} b^{-1} \text{......(6}a\text{)},$$

where the summation is carried out first for the values $\sigma = 0, 1, 2, \dots$, and then for $b = 1, 3, 5, \dots$. Here θ denotes such of the numbers $0, 1, 2, \dots, b-1$, for which integers s with $0 \leq s < b$ can be found satisfying the congruence

$$s^2 \equiv \theta \pmod{b},$$

each value of θ being taken as many times in the sum as values of s can be found. The step to be justified is the rearrangement of the series ($6a$) as a power series in q; *i.e.*, writing

$$\theta + b\sigma = M$$

and summing for b so that the coefficient of $q^M \sqrt{M}$ becomes

$$\sum_{b \text{ odd}}^{\infty} (-1)^{\frac{1}{2}(b-1)} f(b)\, b^{-1}.$$

* *Proceedings of the National Academy of Sciences* (Washington, U.S.A.) vol. 4, 1918.

† See my paper, *T.C.P.S.*

‡ Page 366, *T.C.P.S.*

Now

$$\chi(\omega) = f_0 + f_1 + f_2 + \ldots f_\sigma + \ldots,$$

where

$$f_\sigma = \sum_{b \text{ odd}}^{\infty} (-1)^{\frac{1}{2}(b-1)} q^{\theta + b\sigma} (\theta + b\sigma)^{\frac{1}{2}} b^{-1}.$$

But, if $\sigma > 0$,

$$|f_\sigma| < \sum_b [|q|^{(1+b)\sigma} b^{\frac{1}{2}} (1+\sigma)^{\frac{1}{2}}] < \frac{(1+\sigma)^{\frac{1}{2}} |q|^\sigma}{[1 - |q|^\sigma]^2},$$

since $\theta < b$ and the value of θ cannot be taken b times. Hence if $|q| < 1$, the double series

$$f_1 + f_2 + \ldots + f_\sigma \ldots$$

is absolutely convergent and can be rearranged in ascending powers of q.

The series f_0 is

$$\frac{q}{1} - \frac{2q\sqrt{1}}{3} + \frac{2q\sqrt{1} + 2q^4\sqrt{4}}{5} - \frac{2q\sqrt{1} + 2q^2\sqrt{2} + 2q^4\sqrt{4}}{7} \ldots (6b),$$

of which the sum of $\frac{1}{2}(m+1)$ terms can be written as

$$S = q\left(1 - \frac{2}{3} + \frac{2}{5} \ldots \pm \frac{2}{m}\right)$$

$$+ q^2\sqrt{2}\left\{-\frac{2}{7} \ldots \pm \frac{f(2)}{m}\right\}$$

$$+ q^3\sqrt{3}\left\{-\frac{2}{11} \ldots \pm \frac{f(3)}{m}\right\}$$

....

$$+ q^{m-1}\sqrt{(m-1)}\left\{\ldots \pm \frac{f(m-1)}{m}\right\},$$

where $f(r)$ denotes the number of solutions of

$$s^2 \equiv r \pmod{m}.$$

Now the coefficients of q, $q^2\sqrt{2}$, ... are convergent series* ; calling their sum to infinity a, b, etc., we can write

$$a = 1 - \frac{2}{3} + \frac{2}{5} \ldots \pm \frac{2}{m} + \xi_1,$$

* See § 3.

where $|\xi_1|$ can be made as small as we please by taking m large enough, and similarly for $b, c, \ldots$. Hence, if $n=\frac{1}{2}(m+1)$,

$$|S-aq-bq^2\sqrt{2}\ldots-kq^{n-1}\sqrt{(n-1)}|$$
$$<|\xi|\,[|q|+\sqrt{2}\,|q|^2+\ldots\sqrt{(n-1)}\,|q|^{n-1}]+\eta,$$

where η and $|\xi|$, which is the greatest of $|\xi_1|, |\xi_2|, |\xi_3|, \ldots, |\xi_n|$, can be made as small as we please by taking m large enough. It is obvious then that we can arrange f_0 in a power series for q, of which the coefficient of $\sqrt{M}q^M$ is given by (since $M=\theta$ with $0\leq\theta<b$)

$$\sum_{b>M}(-1)^{\frac{1}{2}(b-1)}f(b)\,b^{-1},$$

where $b=M+1, M+2, M+3, \ldots$ in this order, or rather the odd values in the sequence. Taking now the coefficient of q^M, arising from the other values of σ (which, as already proved, give an absolutely convergent series), only a finite number of values of b arise, *e.g.*, if

$$\sigma=1,\ M=\theta+\ b \text{ so that } b\leq M<2b,$$
$$\sigma=2,\ M=\theta+2b \text{ so that } 2b\leq M<3b,$$

and these values of b give the sum

$$\sum_{b\leq M}(-1)^{\frac{1}{2}(b-1)}f(b)\,b^{-1},$$

so that the coefficient of q^M is as stated.

§ 3. We must show now that

$$A_M\sqrt{M}=F(M),$$

that is

$$\frac{F(M)}{\sqrt{M}}=\frac{f(1)}{1}-\frac{f(3)}{3}+\frac{f(5)}{5}-\ldots\ldots\ldots(7),$$

where $f(n)$ denotes the number of solutions of the congruence

$$\xi^2\equiv M \pmod{n}.$$

Noting that if p and q are both odd and prime to each other, then

$$f(p)f(q)=f(pq),$$
$$(-1)^{\frac{1}{2}(p-1)+\frac{1}{2}(q-1)}=(-1)^{\frac{1}{2}(pq-1)},$$

the series for A_M, whose convergence is as yet undemonstrated, can be formally transformed into the infinite product

$$A_M = \prod_p \left(1 + \frac{(-1)^{\frac{1}{2}(p-1)} f(p)}{p} + \frac{(-1)^{\frac{2}{2}(p-1)} f(p^2)}{p^2} + \frac{(-1)^{\frac{3}{2}(p-1)} f(p^3)}{p^3} + \ldots\right) = \Pi D_p, \text{ say},$$

since

$$(-1)^{\frac{1}{2}(p^r-1)} = (-1)^{\frac{1}{2}(p-1)r},$$

the product referring to the odd primes 3, 5, 7, ...

As will be seen from the following, there is no loss of generality in supposing that M is of the form

$$M = Nq^{2\alpha}s^{2\gamma},$$

where q and s are different odd primes and N has no odd squared factors.

If p is not equal to either q or s,

$$f(p^r) = 1 + \left(\frac{M}{p}\right),$$

where $\left(\frac{M}{p}\right)$ is the Legendre-Jacobi symbol of quadratic reciprocity. Also

$$D_p = 1 + \sum_{r=1}^{\infty} \left[1 + \left(\frac{M}{p}\right)\right] (-1)^{\frac{1}{2}(p-1)r} p^{-r}$$

$$= 1 + \frac{\dfrac{(-1)^{\frac{1}{2}(p-1)}}{p}\left\{1 + \left(\dfrac{M}{p}\right)\right\}}{1 - \dfrac{(-1)^{\frac{1}{2}(p-1)}}{p}}$$

$$= \frac{1 + \dfrac{1}{p}\left(\dfrac{-M}{p}\right)}{1 - \dfrac{(-1)^{\frac{1}{2}(p-1)}}{p}}.$$

The consideration of this infinite product

$$\Pi = \prod_p \frac{1 + \dfrac{1}{p}\left(\dfrac{-M}{p}\right)}{1 - \dfrac{1}{p}\left(\dfrac{-1}{p}\right)},$$

where the product refers to all the odd primes except q or s, shows not only that the series for A_M is convergent, but also that it is equal to ΠD_p. For, writing

$$\prod_p \left[1+\frac{1}{p}\left(\frac{-M}{p}\right)\right] = \Pi\left(1-\frac{1}{p^2}\right)\Big/\Pi\left[1-\frac{1}{p}\left(\frac{-M}{p}\right)\right],$$

we find*

$$\Pi = \Pi\left(1-\frac{1}{p^2}\right) . \Sigma\frac{1}{n}\left(\frac{-1}{n}\right) . \Sigma\frac{1}{n}\left(\frac{-M}{n}\right),$$

where the right-hand Π refers to all terms except q and s, while the summations* refer to odd values of n prime to q and s. These last two series are special cases of well-known Dirichlet's series, and satisfy conditions enabling us to write their product as a Dirichlet's series.† It is then also legitimate‡ to multiply their product by the absolutely convergent Dirichlet's series arising from the product

$$\Pi\left(1-\frac{1}{p^2}\right).$$

This shows that the Dirichlet's series for Π is convergent and equal to the infinite product. Hence this statement also applies to the series and infinite products for A_M, as is seen from the formulæ we shall now give for D_q and D_s.

When p is equal to either q or s the number of solutions of

$$\xi^2 \equiv Nq^{2a}s^{2\gamma} \pmod{p^r}$$

must be found. Taking $p=q$ for example, then when r is even, say $r=2\zeta$,

$$f(q^r) = q^\zeta \text{ if } r \leq 2a,$$

$$f(q^r) = q^a\left\{1+\left(\frac{N}{q}\right)\right\} \text{ if } r > 2a,$$

but when r is odd, say $r=2\zeta+1$,

$$f(q^r) = q^\zeta \text{ if } r \leq 2a,$$

$$f(q^r) = q^a\left\{1+\left(\frac{N}{q}\right)\right\} \text{ if } r > 2a.$$

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* For the equality of the two semi-convergent series in this expression and the corresponding infinite products, see Landau, *Primzahlen*, vol. i., p. 449.

† Landau, *Primzahlen*, vol. ii., p. 685. See also § 6 of this paper.

‡ Landau, *Primzahlen*, vol. ii., p. 671.

Hence the expression for D_q, on taking first $r=0$ and then summing separately for the odd values of $r<2a$, the even values of $r \leq 2a$, and then all the values of $r>2a$, becomes, on putting $\eta=(-1)^{\frac{1}{2}(q-1)}$,

$$D_q = 1 + \frac{\eta}{q} + \frac{q\eta^3}{q^3} + \ldots + \frac{q^{a-1}\eta^{2a-1}}{q^{2a-1}}$$

$$+ \frac{q}{q^2} + \frac{q^2}{q^4} + \ldots \frac{q^a}{q^{2a}}$$

$$+ \sum_{t=1}^{\infty} \frac{q^a\left\{1 + \left(\frac{N}{q}\right)\right\}\eta^t}{q^{2a+t}},$$

and reduces to

$$D_q = 1 + (1+\eta)\left(\frac{1}{q} + \frac{1}{q^2} + \ldots \frac{1}{q^a}\right) + \frac{\eta}{q^{a+1}} \frac{\left[1 + \left(\frac{N}{q}\right)\right]}{1 - \eta/q}.$$

Noting now that

$$\left(1 - \frac{\eta}{q}\right)\left[1 + (1+\eta)\left(\frac{1}{q} + \frac{1}{q^2} + \ldots \frac{1}{q^a}\right)\right]$$

$$= 1 + (1+\eta)\left[\frac{1}{q} + \frac{1}{q^2} + \ldots \frac{1}{q^a}\right]$$

$$- \frac{\eta}{q} - (1+\eta)\left(\frac{1}{q^2} + \frac{1}{q^3} + \ldots \frac{1}{q^{a+1}}\right)$$

$$= 1 + \frac{1}{q} - \frac{(1+\eta)}{q^{a+1}},$$

we have

$$\left(1 - \frac{\eta}{q}\right) D_q = 1 + \frac{1}{q} - \frac{1}{q^{a+1}} + \frac{1}{q^{a+1}}\left(\frac{-N}{q}\right)$$

$$= 1 + \frac{1}{q} - \frac{1}{q^a} - \frac{1}{q^{a+1}} + \frac{1}{q^a}\left\{1 + \frac{1}{q}\left(\frac{-N}{q}\right)\right\}$$

$$= \left(1 + \frac{1}{q}\right)\left(1 - \frac{1}{q^a}\right) + \frac{1}{q^a}\left\{1 + \frac{1}{q}\left(\frac{-N}{q}\right)\right\}.$$

Hence, since

$$\left(1 + \frac{1}{q}\right)\left(1 - \frac{1}{q}\right) = \left(1 + \frac{\eta}{q}\right)\left(1 - \frac{\eta}{q}\right),$$

we have

$$D_q = \frac{1}{q^a} \frac{1+\frac{1}{q}\left(\frac{-N}{q}\right)}{1-\frac{\eta}{q}} + \left(1+\frac{1}{q}+\frac{1}{q^2}+\dots\frac{1}{q^{a-1}}\right)\left(1+\frac{\eta}{q}\right),$$

and similarly for D_s.

Hence A_M can be expressed as the sum of four infinite products, of which the first is

$$\frac{1}{q^a s^\gamma} \Pi \frac{1+\frac{1}{p}\left(\frac{-N}{p}\right)}{1-\frac{(-1)^{\frac{1}{2}(p-1)}}{p}},$$

where Π refers to all the odd primes. Multiplying numerators and denominators by $\Pi\left\{1-\frac{1}{p}\left(\frac{-N}{p}\right)\right\}$, this infinite product becomes

$$\frac{1}{q^a s^\gamma} \Pi \frac{\left(1-\frac{1}{p^2}\right)}{1-\frac{(-1)^{\frac{1}{2}(p-1)}}{p}} \frac{1}{1-\frac{1}{p}\left(\frac{-N}{p}\right)} = \frac{1}{q^a s^\gamma} \frac{2}{\pi} \Sigma \frac{1}{r}\left(\frac{-N}{r}\right),$$

where the summation refers to all odd values of r, and from equation (2) this is

$$\frac{1}{q^a s^\gamma} \frac{f(N)}{\sqrt{N}} = \frac{1}{\sqrt{M}} f(N).$$

The second infinite product of the form is

$$\frac{1}{s^\gamma} \Pi \frac{1+\frac{1}{p}\left(\frac{-N}{p}\right)}{1-\frac{(-1)^{\frac{1}{2}(p-1)}}{p}} \left(1+\frac{1}{q}+\frac{1}{q^2}+\dots\frac{1}{q^{a-1}}\right)\left\{1+\frac{(-1)^{\frac{1}{2}(q-1)}}{q}\right\},$$

where Π refers to all the odd primes except q, and can be reduced to the form

$$\frac{1}{s^\gamma}\left(1+\frac{1}{q}+\dots\frac{1}{q^{a-1}}\right)\frac{2}{\pi}\Big/\Pi\, 1-\frac{1}{p}\left(\frac{-N}{p}\right),$$

where Π refers to all the odd primes except q. This last expression becomes

$$\frac{2}{\pi s^\gamma}\left(1+\frac{1}{q}+\dots\frac{1}{q^{a-1}}\right)\sum_r \frac{1}{r}\left(\frac{-N}{r}\right),$$

where r takes all odd values prime to q. Noting now that

$$f(Nq^{2\lambda}) = \frac{2q^\lambda}{\pi}\sqrt{N}\sum_r \frac{1}{r}\left(\frac{-N}{r}\right),$$

where r takes all odd values prime to q, and taking $\lambda = 1, 2, \ldots, a$ we see that the second infinite product reduces to

$$\frac{1}{\sqrt{M}}\sum_{\lambda=1}^{a} f(Nq^{2\lambda})$$

By symmetry the third infinite product reduces to

$$\frac{1}{\sqrt{M}}\sum_{\mu=1}^{\gamma} f(Ns^{2\mu}).$$

The fourth infinite product reduces to

$$\frac{2}{\pi}\left(1+\frac{1}{q}+\ldots\frac{1}{q^{a-1}}\right)\left(1+\frac{1}{s}+\ldots\frac{1}{s^{\gamma-1}}\right)\sum_r \frac{1}{r}\left(\frac{-N}{r}\right),$$

where r takes all odd values prime to both s and q, and is also equal to

$$\frac{1}{\sqrt{M}}\sum_{\mu=1}^{\gamma}\sum_{\lambda=1}^{a} f(Nq^{2\lambda}s^{2\mu}).$$

Hence it is obvious that the sum of the four infinite products is $\frac{1}{\sqrt{M}} f(M)$, as was stated.

§ 4. We have also other expansions similar to expansions (5) and (5a). Let $G(n)$ be the total number of classes of binary quadratics of determinant $-n$, with the same convention for the forms $(k, 0, k)$ as in § 1, and with the additional conventions that the forms $(2k, k, 2k)$ are reckoned as $\frac{1}{3}$ instead of 1, and that $G(0) = -1/12$. Then the expansions corresponding to (5) and (5a) are

$$\pi\sum_{n=1}^{\infty}[4F(n) - 3G(n)]q^n = \sum_{a,\,b}\frac{i^{-a}\sum\limits_{s=0}^{b-1} e^{-\pi i a s^2/b}}{\sqrt{b}\,[\sqrt{\{-i(a+b\omega)\}}]^3} \quad \ldots(8),$$

$$4\sum_{n=1}^{\infty}[4F(n) - 3G(n)]q^n = \sum_{a,\,b}\frac{e^{-3\pi i a/4}\left(\frac{b}{a}\right)}{[\sqrt{\{-i(a+b\omega)\}}]^3} \quad \ldots\ldots (8a),$$

where the summation is taken first for all odd values of a, positive or negative, and then for all positive even values of b,

excluding $b=0$, while the meaning of the symbol $\left(\frac{b}{a}\right)$ is as in equation (5a). The difficulty as to the order of summation in equation (8a) is of the same kind as in equation (5a). The right-hand side of (8), by summing with respect to a,* can be written as

$$2\pi \sum_1^\infty B_M M^{\frac{1}{2}} q^M,$$

where
$$B_M = \sum_{b \text{ even}} \frac{\phi(b)(-1)^{(s^2-M+\frac{1}{2}b)/b}}{b},$$

and $\phi(b)$ denotes the number of solutions of

$$s^2 \equiv M - \tfrac{1}{2}b \pmod{b} \qquad \text{(9)},$$

and b takes all even positive values. The justification of the re-arrangement by which we get the term q^M is similar to that occurring for equation (5) and need not be repeated. We have now to prove the equation corresponding to equation (7), namely,

$$\frac{4F(M)-3G(M)}{2\sqrt{M}} = \sum_{b \text{ even}} \frac{\phi(b)}{b}(-1)^{(s^2-M+\frac{1}{2}b)/b} \qquad \text{(10)}.$$

In this sum, when M is not divisible by 4, there is no need to consider values of b divisible by 8. For, writing $b=8\beta$, equation (9) shows that s must be odd so that the solutions of (9) can be grouped in pairs such as s, $4\beta-s$ which are different. Hence

$$\frac{s^2-M+4\beta}{8\beta} \equiv \frac{(4\beta-s)^2-M+4\beta}{8\beta}+1 \pmod{2},$$

so that the sum (10) vanishes for the values of b which are divisible by 8.

Remembering now that

$$\begin{aligned} G(M) &= F(M) \text{ if } M \equiv 1, 2 && \pmod{4}, \\ 3G(M) &= 4F(M) \text{ if } M \equiv 3 && \pmod{8}, \\ G(M) &= 2F(M) \text{ if } M \equiv 7 && \pmod{8}, \end{aligned}$$

the truth of (10) easily follows for these values of M from equation (7). Thus suppose $M \equiv 3 \pmod{8}$ and consider first the case when b is twice an odd number, say 2β. Then

$$s^2 \equiv M - \beta \pmod{2\beta},$$

* See my paper, *T.C.P.S.*, pp. 367, 368.

so that s is even and

$$\frac{s^2 - M + \beta}{2\beta} \equiv \frac{\beta - 3}{2} \pmod 2.$$

Since there is a one-to-one correspondence between the solution of $s^2 \equiv M \pmod \beta$ and $s^2 \equiv M - \beta \pmod{2\beta}$,

$$\phi(2\beta) = f(\beta).$$

Hence the series (10) reduces to $-\frac{1}{2}F(M)$.

When b is four times an odd number, say 4β, we have

$$s^2 \equiv M - 2\beta \pmod{4\beta},$$

so that s is odd and

$$\frac{s^2 - M + 2\beta}{4\beta} \equiv \frac{\beta - 1}{2} \pmod 2.$$

Also
$$\phi(4\beta) = 2f(\beta),$$

so that in this case the series (10) reduces to $\frac{1}{2}F(M)$. Hence the total sum of the series (10) is zero and agrees with the left-hand side of (10).

When M is divisible by 4, but not by 16 (the general case when M is divisible by any power of 4 follows at once by induction), we note that

$$F(4n) = 2F(n) \text{ and } G(4n) = G(n) + F(4n),$$

so that, on replacing M by $4M$, equation (10) becomes

$$\frac{2F(M) - 3G(M)}{4\sqrt{M}} = \sum_{b \text{ even}} \frac{\phi(b)}{b} (-1)^{(s^2 - 4M + \frac{1}{2}b)/b} \quad \text{...(11)},$$

where
$$s^2 \equiv 4M - \tfrac{1}{2}b \pmod b$$

and M is not now divisible by 4. This congruence shows that b is twice an odd number, say 2β, with β odd, or a multiple of 8, say 4β, where β is even. In the first case s is odd and

$$(s^2 - 4M + \tfrac{1}{2}b)/b \equiv \tfrac{1}{2}(\beta + 1) \pmod 2,$$

so that the corresponding part of (11) becomes

$$-\frac{1}{2}\frac{F(4M)}{\sqrt{(4M)}} = -\frac{F(M)}{2\sqrt{M}}.$$

When b is divisible by 8, s must be even, say 2σ, so that

$$\sigma^2 \equiv M - \tfrac{1}{2}\beta \pmod \beta$$

and
$$\phi(b) = 2\phi(\beta),$$

since the number of solutions of

$$s^2 \equiv 4M - 2\beta \pmod{4\beta}$$

is twice the number of solutions of

$$\sigma^2 \equiv M - \tfrac{1}{2}\beta \pmod{\beta}.$$

Hence the corresponding part of (11) becomes half of the series (10); that is

$$\frac{4F(M) - 3G(M)}{4\sqrt{M}},$$

and hence the whole series (11) reduces to

$$\frac{2F(M) - 3G(M)}{4\sqrt{M}},$$

as was to be shewn.

§ 5. Though I think the expressions (7) and (10) for $F(n)$ and $G(n)$ are novel, it should be noticed that they are really special cases of general results in the arithmetical theory of the general quadratic form. In the particular case of a definite form, however, it is necessary to introduce a term called the weight of the form, which is defined as the reciprocal of the number of linear substitutions with integral coefficients and determinant positive unity which change the form into itself. The weight of a complex of forms is then defined as the sum of the weights of the non-equivalent forms of the complex, and is a more fundamental notion than the number of classes of forms of the complex, as the weight must be known before we can find the number of classes*.

The conventions adopted in §§ 1 and 4 are such that $F(M)$ $G(M)$ are practically the weights of certain complexes of binary forms. But expressions for the weights of the complex of forms included in a given genus have been given by H. J. S. Smith and Minkowski†, and the formulæ (7) and (10) can be considered as special cases of these results in the case of quadratic forms with two variables.

§ 6. The multiplication theorem quoted in § 3 (*i.e.* Landau, vol. ii., p. 685) states that if

$$a_n = O(1/n),$$

$$A(x) = \sum_{n=1}^{x} a_n = A + o(1/\log x),$$

* See my paper, "On the class number for definite ternary quadratics", *Messenger of Mathematics*, vol. xlvii. (1917), p. 65.

† See Bachmann, *Zahlentheorie*, vol. iv., chap. 10; especially Minkowski's form of the result on page 624.

and
$$b_n = O(1/n),$$
$$B(x) = \sum_{n=1}^{x} b_n = B + o(1/\log x),$$
then
$$AB = \sum_{1}^{\infty} c_n,$$
where
$$c_n = \sum_{l|n} a_l b_{n/l},$$
and $\sum_{l|n}$ refers to all the divisors l of n.

This theorem enables me to prove the validity of equation (5a) [and also of equation (8a)] as follows. Noting equation (6a) we can write (5) in the form

$$2\pi\chi(\omega) = f(1) + f(3) + f(5) + f(7) \ldots,$$

where

$$f(b) = \sum_a \frac{(-1)^{\frac{1}{2}(b-1)} \sum_{s=0}^{b-1} e^{-\pi i a s^2/b}}{\sqrt{b}\,[\sqrt{\{-i(a+b\omega)\}}]^3}, \quad a = 0, \pm 2, \pm 4, \ldots$$
$$= 2\pi \sum_\sigma (-1)^{\frac{1}{2}(b-1)} q^{\theta+b\sigma} (\theta + b\sigma)^{\frac{1}{2}} b^{-1}, \quad \sigma = 0, 1, 2, 3, \ldots,$$

where θ is as in (6a). Also from

$$\tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \ldots,$$

we find (Landau, vol. ii., p. 676)

$$\frac{4}{\pi} = 1 + \tfrac{1}{3} - \tfrac{1}{5} + \ldots + \frac{\mu(n)(-1)^{\frac{1}{2}(n-1)}}{n} \ldots,$$

where $\mu(n)$, the well-known arithmetical function of n, is zero if n is divisible by a squared factor, is 1 if $n=1$, and $(-1)^\rho$ if n is the product of ρ primes.

To prove equation (5a) we simply have to justify the formal multiplication (in the way given at the beginning of this section) of the series for $2\pi\chi(\omega)$ and $4/\pi$, as it is easily verified that [as ought to be the case from the method of finding (5a)]

$$\sum_{mn=b} \frac{\mu(n)(-1)^{\frac{1}{2}(n-1)}}{n} f(m) = \sum_{a \text{ even}}^{\pm\infty} \frac{i^{-\frac{1}{2}(b-1)} \dfrac{a}{b}}{[\sqrt{\{-i(a+b\omega)\}}]^3}.$$

This follows [because $f(m)$ is an absolutely convergent series when summed for a] by putting* $a = dA$, $m = dM$, where d is

* Cf. *T.C.P.S.*, p. 365.

the greatest common divisor of a and m, and noting the well-known result

$$\sum_{d|N} \mu(d) = 0 \text{ or } 1,$$

according as N is not or is equal to 1. The left-hand side then becomes $f(b)$, omitting those terms in $f(b)$ for which a is not prime to b.

The series for $4/\pi$ is of the type indicated by Σb_n (Landau, vol. ii., p. 692). Also, if θ is as in (6a),

$$f(b)/2\pi = \sum_{\theta} (-1)^{\frac{1}{2}(b-1)} q^{\theta} \theta^{\frac{1}{2}b-1} + \sum_{\sigma=1}^{\infty} (-1)^{\frac{1}{2}(b-1)} q^{\theta+b\sigma} (\theta + b\sigma)^{\frac{1}{2}b-1}.$$

The second series is clearly $O(q^{\epsilon b})$, where ϵ is small, while the first series is simply the general term in equation (6b) and is obviously $O(1/b)$. Hence $f(b) = O(1/b)$. It is also clear from (6b) that

$$f(b+1) + f(b+2) + \ldots = O(1/b) = o(1/\log b).$$

Hence the series $\Sigma f(b)$ is of the type indicated by Σa_n. This proves the validity of equation (5a), which it may be noted can be written in the more elegant form*

$$8\sum_{1}^{\infty} F(n)\, q^n = \Sigma \left\{ \frac{\theta_{00}(\omega)}{\theta_{00}\left(\dfrac{c+d\omega}{a+b\omega}\right)} \right\}^3,$$

where the summation is taken first for $a = 0, \pm 2, \pm 4, \ldots$; and then for $b = 1, 3, 5, \ldots$ in this order. Also d is any even integer and c any odd integer satisfying $ad - bc = 1$, so that the general term is independent of c and d, while

$$\theta_{00}(\omega) = 1 + 2q + 2q^4 + 2q^9 + \ldots, \quad q = e^{\pi i \omega}.$$

Similarly equation (9a), which from the above is clearly valid, can be written as

$$\sum_{1}^{\infty} [16F(n) - 12G(n)]\, q^n = \Sigma \left\{ \frac{\theta_{00}(\omega)}{\theta_{00}\left(\dfrac{c+d\omega}{a+b\omega}\right)} \right\}^3,$$

where the summation is taken first for all odd positive or negative values of a; and then for all $b = 2, 4, 6, \ldots$ in this order. Now, however, d is odd and c is even, while as before $ad - bc = 1$.

* Cf. *T.C.P.S.*, p. 363.

THE EXPRESSION OF BESSEL FUNCTIONS OF POSITIVE ORDER AS PRODUCTS, AND OF THEIR INVERSE POWERS AS SUMS OF RATIONAL FRACTIONS.

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1. An important theorem in complex integration, originally due to Cauchy*, leads in certain cases of fairly wide range to the expansion of uniform meromorphic functions in series of rational fractions; and applications of the theorem have been made by Mittag-Leffler†, Picard‡, Goursat§, Borel||, Lindelöf¶, and others. The purpose of this paper is to obtain, by means of the theorem in question, some results connected with Bessel functions of integral order; but it is clear from the analysis that applications can be made similarly to such functions of merely positive order, and one example is given.

The theorem can be stated** as follows:

Let $F(z)$ be a uniform meromorphic function, having isolated poles and no essential singularities in the finite part of the z-plane. In the immediate vicinity of a pole a_n, let the polar part of $F(z)$ be

$$G_n\left(\frac{1}{z-a_n}\right)=\frac{A_s}{(z-a_n)^s}+\frac{A_{s-1}}{(z-a_n)^{s-1}}+\ldots+\frac{A_1}{z-a_n};$$

and suppose that the function is regular near the origin. Further, let complete contours C_m enclosing the origin be drawn, gradually increasing in magnitude so as to enclose poles in succession but not to pass through any pole; and let them be such that along C_m, as $m \rightarrow \infty$,

$$|z^{-p}F(z)|$$

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* *Œuvres complètes de Cauchy*, 2. Sér., t. vii, pp. 324 *et seq.*

† *Acta Soc. Fenn.*, t. xi. (1880), pp. 273—293.

‡ *Traité d'Analyse*, t. ii., ch. vi.

§ *Cours d'Analyse*, t. ii., ch. xv.

|| *Leçons sur les fonctions méromorphes*, ch. iv.

¶ *Le calcul des résidus*, ch. ii.

** It is used in my *Theory of Functions*, § 61: but, there, the boundary is taken to be circular.

remains finite (usually it is made to tend uniformly to zero), p being the smallest positive integer for which the condition is satisfied. Finally, along this contour, denoting by δ the smallest value of $|z|$ and by l_m the length of C_m, let l_m/δ be finite. When all these conditions are satisfied, we have

$$F(x) = F(0) + xF'(0) + \ldots + \frac{x^p}{p!} F^{(p)}(0)$$
$$+ \lim_{m\to\infty} \sum_{n=1}^{m} \left\{ G_n\left(\frac{1}{x-a_n}\right) + S_{n,0} + xS_{n,1} + \ldots + x^p S_{n,p} \right\},$$

where $S_{n,\mu}$ is the residue of $z^{-\mu-1}F(z)$ for the pole a_n.

Manifestly, the results and the conditions are of an extensive character. To use the theorem, it is necessary to know the poles of $F(z)$ and the polar part of $F(z)$ near any pole: to determine a finite integer p if it exists: and to secure the finiteness in the value of l_m/δ.

2. The following applications are made to meromorphic functions, arising through Bessel functions, such as $J_1(z)/J_0(z)$, $1/J_0(z)$, and so on. We need the zeros of the Bessel functions of positive integral order. It is known that for $J_n(z)$, where n is a positive integer, $z=0$ is a zero of order n, while all the remaining zeros are real, simple, and associable in pairs with equal and opposite signs. They are most easily derivable from the asymptotic expression for $J_n(z)$, which is

$$J_n = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left\{1 - \frac{(1^2-4n^2)(3^2-4n^2)}{2!\,(8z)^2} + \ldots\right\} \cos\left(z - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right)$$
$$+ \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left\{\frac{1^2-4n^2}{8z} - \frac{(1^2-4n^2)(3^2-4n^2)(5^2-4n^2)}{3!\,(8z)^3} + \ldots\right\}$$
$$\times \sin\left(z - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right).$$

When we deal specially with the roots of $J_0(z)$, we note that the most important term in this asymptotic expression for $J_0(z)$ is

$$\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \tfrac{1}{4}\pi\right),$$

so that, for large values of the roots, a first approximation to the p^{th} positive root in ascending order of magnitude is

$$\left(p - \tfrac{1}{4}\right)\pi.$$

The earlier positive roots have been calculated*; and, even at the beginning of the succession, they approximate fairly

* Willson and Peirce, *Bull. Amer. Math. Soc.*, t. iii. (1897), p. 153–155.

to $\frac{3}{4}\pi$, $\frac{7}{4}\pi$, $\frac{11}{4}\pi$, ... For large values of p, and using not merely the one term in the asymptotic expression, we write

$$p - \tfrac{1}{4} = p':$$

and we find that the p^{th} root κ_p is given by

$$\kappa_p = p'\pi + \frac{1}{8p'\pi} - \frac{124}{3(8p'\pi)^3} - \dots .$$

Manifestly, the series

$$\sum_{p=1}^{\infty} \frac{1}{\kappa_p}$$

diverges, while the series

$$\sum_{p=1}^{\infty} \frac{1}{\kappa_p^2}$$

converges; and so the function $J_0(z)$ is of class unity, in the sense of the word class as introduced by Laguerre*, in so far as the zeros are concerned.

Moreover, the roots of $J_0(z)$ are equal and opposite in sign: consequently, a function having all the roots of J_0 simple, as they are simple for J_0, is the absolutely and uniformly converging product

$$\prod_{-\infty}^{p=\infty} \left\{\left(1 - \frac{z}{\kappa_p}\right) e^{z/\kappa_p}\right\}.$$

It follows therefore, from the Weierstrass theory of such products of primary factors, that $J_0(z)$ is of the form

$$e^{I(z)} \prod_{-\infty}^{p=\infty} \left\{\left(1 - \frac{z}{\kappa_p}\right) e^{z/\kappa_p}\right\},$$

where $I(z)$ is an integral function of z, which must be even because $J_0(z)$ is an even function. It will be proved that $I(z)$ is zero, by means of the Cauchy theorem already quoted.

3. Next, as regards the zeros of $J_n(z)$, where n is a positive integer greater than zero, we again note the most important term in the asymptotic expression for $J_n(z)$, viz.

$$\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right);$$

and we obtain a first approximation to the p^{th} root in succession after $z = 0$ (which is a zero of order n) in the form

$$\left(p + \tfrac{1}{2}n - \tfrac{1}{4}\right)\pi.$$

* *Œuvres de Laguerre*, t. i., pp. 171–180; my *Theory of Functions*, §§ 59–61.

Writing
$$q = (p + \tfrac{1}{2}n - \tfrac{1}{4}),$$
$$a_1 = 1^2 - 4n^2,$$
$$a_2 = (1^2 - 4n^2)(3^2 - 4n^2),$$
$$a_3 = (1^2 - 4n^2)(3^2 - 4n^2)(5^2 - 4n^2),$$
and so on, and denoting the root in question λ_p, we find

$$\lambda_p = q\pi + \frac{a_1}{89\pi} - \frac{1}{3\,!}\frac{1}{(8q\pi)^3}(a_3 - 3a_1a_2 + 2a_1^3 + 48a_1) - \dots .$$

From this expression for the roots of J_n, it is clear that no root of any function J of positive integral order is equal to a root of any other function J also of positive integral order.

The zeros* of J_n other than $z=0$ are equal and opposite in pairs. Clearly the series

$$\sum_{p=1}^{\infty} \frac{1}{\lambda_p}$$

diverges, while the series

$$\sum_{p=1}^{\infty} \frac{1}{\lambda_p^2}$$

converges. A use of Cauchy's theorem, corresponding to the use for the function $J_0(z)$, will lead to the result that

$$J_n(z) = \frac{z^n}{2^n \Pi\,n)} \prod_{p=+\infty}^{\infty} \left\{\left(1 - \frac{z}{\lambda_p}\right) e^{z/\lambda_p}\right\},$$

where the product converges uniformly and absolutely.

When the primary factors corresponding to equal and opposite roots are combined, we have

$$J_0(z) = \prod_{p=1}^{\infty} \left(1 - \frac{z^2}{\kappa_p^2}\right),$$

$$J_n(z) = \frac{z^n}{2^n \Pi(n)} \prod_{p=1}^{\infty} \left(1 - \frac{z^2}{\lambda_p^2}\right).$$

4. In considering the application of Cauchy's theorem, we shall require the order of the value of $|J_0(z)|$, and of

$$|J_{n+1}(z) \div J_n(z)|,$$

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* For references to authorities concerning the zeros of the functions $J_n(z)$, a convenient summary will be found in Nielsen, *Handbuch d. Theorie d. Cylinderfunktionen*, ch. xi.

for large values of $|z|$ along a contour. As this contour, we choose a square, centre the origin, having its sides parallel to the axis, and having one side passing through the point $x = m\pi$, $y = 0$, where the integer $m \rightarrow \infty$ in a later limit. In this case* the length of the contour is $8m\pi$, and, along the contour, the quantity $|z| \geq m\pi$; so that the magnitude l_m/δ, required in the use of the theorem, is 8 and therefore is finite. Taking the asymptotic expression of the functions, an adequate estimate will be obtained as to finiteness, if we choose the first term: so that, for large values of $|z|$, we can take

$$|J_n| \sim \left| \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \frac{\pi}{4} - n\frac{\pi}{2}\right) \right|.$$

Thus, when n is even,

$$\begin{aligned} |J_n| &\sim \left| \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \frac{\pi}{4}\right) \right| \\ &\sim \frac{1}{(\pi|z|)^{\frac{1}{2}}} |\cos z + \sin z| \\ &\sim \frac{1}{(\pi|z|)^{\frac{1}{2}}} (\cosh 2y + \sin 2x)^{\frac{1}{2}}; \end{aligned}$$

and, when n is odd,

$$|J_n| \sim \frac{1}{(\pi|z|)^{\frac{1}{2}}} (\cosh 2y - \sin 2x)^{\frac{1}{2}}.$$

Hence
$$\left|\frac{J_1}{J_0}\right| \sim \left(\frac{\cosh 2y - \sin 2x}{\cosh 2y + \sin 2x}\right)^{\frac{1}{2}}.$$

Now along the sides of the square parallel to the axis of y, $x = m\pi$ or $-m\pi$: thus

$$\left|\frac{J_1}{J_0}\right| \sim 1$$

along those sides. Along the sides parallel to the axis of x, x is variable and $y = \pm m\pi$, where m is large; thus

$$\left|\frac{J_1}{J_0}\right| \sim \left| \frac{1 - \dfrac{\sin 2x}{\cosh 2y}}{1 + \dfrac{\sin 2x}{\cosh 2y}} \right| \sim 1$$

along those sides. Consequently

$$\left|\frac{1}{z}\frac{J_1}{J_0}\right|$$

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* It is the contour chosen by Goursat for the function $\cot z - 1/z$; *Cours d'Analyse*, t. ii., p. 164.

tends uniformly to the value zero along the contour; and therefore for the meromorphic function $J_1(z)/J_0(z)$, the integer p is unity.

Again, the square includes as poles of this meromorphic function the zeros $\pm\kappa_1, \pm\kappa_2, \ldots, \pm\kappa_n$ of the function $J_0(z)$. Now near a root κ, we have

$$J_0(z) = J_0(\kappa) + \theta J_0'(\kappa) + \ldots$$
$$= -\theta J_1(\kappa),$$

and therefore the residue of

$$\frac{J_1(z)}{J_0(z)}$$

is -1; consequently

$$G_n\left(\frac{1}{z-\kappa_n}\right) = -\frac{1}{z-\kappa_n}.$$

The residue of
$$\frac{1}{z}\frac{J_1(z)}{J_0(z)}$$

is similarly $-1/\kappa_n$ for the root κ_n and is $+1/\kappa_n$ for the root $-\kappa_n$; hence $S_{n,0}$ is $-1/\kappa_n$ for the root κ_n and is $+1/\kappa_n$ for the root $-\kappa_n$. Moreover

$$\frac{J_1(z)}{J_0(z)}$$

is zero when $z=0$; hence

$$\frac{J_1(z)}{J_0(z)} = \underset{m\to-\infty}{\sum_{-m}^{m}} \left(-\frac{1}{z-\kappa_n} - \frac{1}{\kappa_n}\right),$$

and the right-hand side converges uniformly as $m \to \infty$; that is,

$$\frac{J_1(z)}{J_0(z)} = -\sum_{n=-\infty}^{\infty}\left(\frac{1}{z-\kappa_n} + \frac{1}{\kappa_n}\right).$$

But
$$J_1(z) = -\frac{dJ_0}{dz},$$

$$\frac{d}{dz}\log\left\{\left(1-\frac{z}{\kappa_n}\right)e^{z/\kappa_n}\right\} = \frac{1}{z-\kappa_n} + \frac{1}{\kappa_n};$$

and therefore $J_0(z) = A \prod_{-\infty}^{\infty}\left\{\left(1-\frac{z}{\kappa_n}\right)e^{z/\kappa_n}\right\}$,

where A is a constant. Taking $z=0$, we have $A=1$; and therefore

$$J_0(z) = \prod_{-\infty}^{\infty}\left\{\left(1-\frac{z}{\kappa_n}\right)e^{z/\kappa_n}\right\},$$

and also
$$J_0(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\kappa_n^2}\right).$$

As
$$J_0(z) = 1 - \left(\frac{x}{2}\right)^2 \frac{1}{(1\,!)^2} + \left(\frac{x}{2}\right)^4 \frac{1}{(2\,!)^2} + \dots,$$

we at once have
$$\sum_1^{\infty} \frac{1}{\kappa_n^2} = \tfrac{1}{4},$$
$$\sum_1^{\infty} \frac{1}{\kappa_n^4} = \tfrac{1}{32},$$
$$\sum_1^{\infty} \frac{1}{\kappa_n^6} = \tfrac{1}{192},$$

and so on: where the quantities κ in the summations are the positive roots of the equation $J_0(\theta) = 0$.

5. Next, consider the magnitude of
$$\left|\frac{J_{n+1}}{J_n}\right|$$
round the same contour; we have
$$\left|\frac{J_{n+1}}{J_n}\right| \sim \left(\frac{\cosh 2y \mp \sin 2x}{\cosh 2y \pm \sin 2x}\right)^{\frac{1}{2}},$$
where the upper signs are taken when n is even and the lower signs are taken when n is odd.

After the preceding explanation,
$$\left|\frac{J_{n+1}}{J_n}\right| \sim 1$$
along the contour, and then
$$\left|\frac{1}{z}\frac{J_{n+1}}{J_n}\right|$$
tends uniformly to zero along the contour as $m \to \infty$: and therefore the integer $p = 1$.

Denoting any one of the roots by λ, we have
$$J_n(\lambda + \theta) = J_n(\lambda) + \theta J_n'(\lambda) + \dots.$$

Now, for all values of z,
$$2J_n' = J_{n-1} - J_{n+1},$$
$$2nJ_n = z\,(J_{n-1} + J_{n+1});$$

hence, when $z=\lambda$,

$$J_{n-1}=-J_{n+1}, \quad J_n'=-J_{n+1};$$

consequently, for the function $J_{n+1}(z)/J_n(z)$, we have

$$G\left(\frac{1}{z-\lambda}\right)=-\frac{1}{z-\lambda}.$$

Further, the residue of

$$\frac{1}{z}\frac{J_{n+1}(z)}{J_n(z)},$$

for the root $z=\lambda$, is $-\frac{1}{\lambda}$. And when $z=0$, the function $J_{n+1}(z)/J_n(z)$ vanishes. Hence

$$\frac{J_{n+1}(z)}{J_n(z)}=-\sum_{-\infty}^{p=\infty}\left(\frac{1}{z-\lambda_p}+\frac{1}{\lambda_p}\right).$$

Now
$$\frac{d}{dx}\left(\frac{J_n}{x_n}\right)=-\frac{J_{n+1}}{x^n},$$

$$\frac{d}{dx}\log\left\{\left(1-\frac{x}{\lambda_p}\right)e^{x/\lambda_p}\right\}=\frac{1}{x-\lambda_p}+\frac{1}{\lambda_p};$$

consequently
$$\frac{J_n(x)}{x^n}=C\prod_{-\infty}^{p=\infty}\left\{\left(1-\frac{x}{\lambda_p}\right)e^{x/\lambda_p}\right\}.$$

The first term in the customary expansion of J_n is

$$\left(\frac{x}{2}\right)^n\frac{1}{\Pi(n)};$$

and therefore

$$J_n(x)=\frac{x^n}{2^n\Pi(n)}\prod_{-\infty}^{p=\infty}\left\{\left(1-\frac{x}{\lambda_p}\right)e^{x/\lambda_p}\right\}.$$

When the primary factors corresponding to equal and opposite roots are combined, we have

$$J_n(x)=\frac{x^n}{2^n\Pi(n)}\prod_{p=1}^{\infty}\left(1-\frac{x^2}{\lambda_p^2}\right).$$

It follows, from the coefficients in the power series for $J_n(x)$, that

$$\sum_1^\infty\frac{1}{\lambda_p^2}=\frac{1}{4(n+1)},$$

$$\sum_1^\infty\frac{1}{\lambda_p^4}=\frac{1}{16(n+1)^2(n+2)},$$

$$\sum_1^\infty\frac{1}{\lambda_p^6}=\frac{1}{32(n+1)^3(n+2)(n+3)},$$

and so on.

6. The preceding result as regards J_n has been derived on the supposition that n is a positive integer; it can be established on the supposition that n is merely a real positive number. The only difference is that the position of the contour has to be changed. We keep it a square, centre the origin, and having its sides parallel to the axes of reference in the z-plane; and writing

$$a = \tfrac{1}{2}n\pi,$$

we make a side of the contour pass through the point

$$z = m\pi + a,$$

where m is a positive integer. Then, along the contour,

$$|J_n| \sim \left|\left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos\left(z - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right)\right|$$

$$\sim \left|\left(\frac{1}{\pi z}\right)^{\frac{1}{2}}\right| \left|\cos(z-a) + \sin(z-a)\right|$$

$$\sim \left|\left(\frac{1}{\pi z}\right)^{\frac{1}{2}}\right| \{\cosh 2y + \sin 2(x-a)\}^{\frac{1}{2}},$$

while $$|J_{n+1}| \sim \left|\left(\frac{1}{\pi z}\right)^{\frac{1}{2}}\right| \{\cosh 2y - \sin 2(x-a)\}^{\frac{1}{2}}.$$

As before, $$\left|\frac{1}{z}\frac{J_{n+1}}{J_n}\right|$$

tends uniformly to zero as $m \to \infty$. Also J_{n+1}/J_n is a uniform meromorphic function, and $F(0)$ is zero; hence the theorem applies. The analysis now is formally the same as before; and we have

$$J_n = \frac{z^n}{2^n \Pi(n)} \prod_{-\infty}^{\infty} \left\{\left(1 - \frac{z}{\rho_q}\right) e^{z/\rho_q}\right\}$$

$$= \frac{z^n}{2^n \Pi(n)} \prod_{q=1}^{\infty} \left(1 - \frac{z^2}{\rho_q^2}\right),$$

where the quantities $\rho_1, \rho_2, \ldots$ are the positive non-zero roots of J_n.

As an immediate verification, let $n = \frac{1}{2}$. Then

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x.$$

The non-zero positive roots of $J_{\frac{1}{2}}(x)$ are $\pi, 2\pi, \ldots$; and so

$$\frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}}\Pi\left(\frac{1}{2}\right)} \prod_{q=1}^{\infty} \left(1 - \frac{x^2}{q^2\pi^2}\right) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x.$$

But $\Pi\left(\frac{1}{2}\right)=\frac{1}{2}\pi^{\frac{1}{2}}$; and so we fall back upon the customary expression for $\sin x$.

Next, consider $J_{\frac{3}{2}}(x)$. The positive non-zero roots, being the positive roots of

$$\tan z = z,$$

are known, from the asymptotic expression for $J_{\frac{3}{2}}(z)=0$, to be

$$\sigma_q = l\pi - \frac{1}{l\pi} + \frac{11}{24}\frac{1}{(l\pi)^3} + \dots,$$

where $$l = q + \tfrac{1}{2},$$

with $q = 1, 2, \dots$. (The quantity a_1 of § 3 is now -8, and other quantities a vanish.) Hence

$$J_{\frac{3}{2}} = \frac{z^{\frac{3}{2}}}{3\left(\frac{1}{2}\pi\right)^{\frac{1}{2}}}\prod_{q=1}^{\infty}\left(1-\frac{z^2}{\sigma_q^{\,2}}\right);$$

that is, $$\frac{\sin z}{z} - \cos z = \frac{z^2}{3}\prod_{q=1}^{\infty}\left(1-\frac{z^2}{\sigma_q^{\,2}}\right).$$

Thus $$\Sigma\frac{1}{\sigma_q^{\,2}} = \frac{1}{10},$$

$$\Sigma\frac{1}{\sigma_q^{\,4}} = \frac{1}{350},$$

and so on.

7. We proceed to consider inverse powers of the Bessel functions, beginning with J_0. The moduli of

$$\frac{1}{zJ_0},\quad \frac{1}{z^2J_0^{\,2}},\quad \frac{1}{z^2J_0^{\,3}},$$

all tend uniformly to zero for large values of z along the square contour previously selected (§ 4); thus the integer p is unity for the function $\frac{1}{J_0}$, and it is two for the functions $\frac{1}{J_0^{\,2}}$, $\frac{1}{J_0^{\,3}}$. Also l_m/δ is equal to 8 as before, while J_0^{-m}, for any positive integer m, is regular in the vicinity of the origin. Thus Cauchy's theorem applies.

In order to find the functions $G_q\left(\frac{1}{z-\kappa_q}\right)$, we need the expansions of the functions in the vicinity of each root κ. Take $z=\kappa+\theta$, where θ is small; then

$$J_0(\kappa+\theta) = -J_1\left\{\theta - \frac{\theta^2}{2\kappa} - \left(1-\frac{2}{\kappa^2}\right)\frac{\theta^3}{3!} + \left(\frac{2}{\kappa}-\frac{6}{\kappa^3}\right)\frac{\theta^4}{4!} - \dots\right\},$$

and consequently

$$\frac{1}{J_0(z)} = -\frac{1}{\theta J_1} + \text{positive powers.}$$

Hence, for $1/J_0(z)$, we have

$$G\left(\frac{1}{z-\kappa}\right) = -\frac{1}{z-\kappa}\frac{1}{J_1(\kappa)}.$$

The residue of $\{zJ_0(z)\}^{-1}$ for the root κ is the residue of

$$\frac{1}{\kappa+\theta}\frac{1}{J_0(\kappa+\theta)},$$

that is, it is

$$-\frac{1}{\kappa}\frac{1}{J_1(\kappa)}.$$

The value of $1/J_0(z)$ at the origin is unity; hence Cauchy's theorem gives

$$\frac{1}{J_0(x)} = 1 - \sum_{-\infty}^{\infty}\frac{1}{J_1(\kappa)}\left(\frac{1}{x-\kappa}+\frac{1}{\kappa}\right).$$

Grouping together the terms for the two roots $\pm\kappa$, we have

$$\frac{1}{J_1(\kappa)}\left(\frac{1}{x-\kappa}+\frac{1}{\kappa}\right) - \frac{1}{J_1(\kappa)}\left(\frac{1}{x+\kappa}-\frac{1}{\kappa}\right)$$

$$= \frac{2}{J_1(\kappa)}\left(\frac{\kappa}{x^2-\kappa^2}+\frac{1}{\kappa}\right);$$

and therefore

$$\frac{1}{J_0(x)} = 1 - 2\sum_{q=1}^{\infty}\left\{\frac{\kappa_q}{x^2-\kappa_q^2}+\frac{1}{\kappa_q}\right\}\frac{1}{J_1(\kappa_q)}.$$

But

$$\frac{1}{J_0(x)} = 1 + \frac{1}{4}x^2 + \frac{3}{64}x^4 + \frac{19}{9.256}x^6 + \dots$$

for sufficiently small values of x, necessarily such that $|x| < \kappa_1$; and so

$$\sum_{q=1}^{\infty}\frac{1}{\kappa_q^{\,3}J_1(\kappa_q)} = \frac{1}{8},$$

$$\sum_{q=1}^{\infty}\frac{1}{\kappa_q^{\,5}J_1(\kappa_q)} = \frac{3}{128},$$

$$\sum_{q=1}^{\infty}\frac{1}{\kappa_q^{\,7}J_1(\kappa_q)} = \frac{19}{9.512},$$

while, generally, the value of

$$\sum_{q=1}^{\infty}\frac{1}{\kappa_q^{\,2m+1}J_1(\kappa_q)}$$

is the coefficient of x^{2m} in the expansion of

$$\left\{1-\left(\frac{x}{2}\right)^2\frac{1}{(1\,!)^2}+\left(\frac{x}{2}\right)^4\frac{1}{(2\,!)^2}-\left(\frac{x}{2}\right)^6\frac{1}{(3\,!)^2}+\ldots\right\}^{-1}$$

in ascending powers of x.

8. In the same way, we have

$$\frac{1}{J_0^2(z)}=\frac{1}{\theta^2 J_1^2}+\frac{1}{\theta}\,\frac{1}{\kappa J_1^2}+\ldots,$$

and therefore

$$G\left(\frac{1}{z-\kappa}\right)=\frac{1}{J_1^2}\left\{\frac{1}{(z-\kappa)^2}+\frac{1}{\kappa\,(z-\kappa)}\right\}.$$

The residue of

$$\frac{1}{z}\,\frac{1}{J_0^2(z)}$$

for the root κ is zero; and the residue of

$$\frac{1}{z^2}\,\frac{1}{J_0^2(z)}$$

for that root is

$$-\frac{1}{\kappa^3 J_1^2}.$$

Moreover, we have

$$\left\{\frac{1}{J_0^2(z)}\right\}_{z=0}=1,\quad \left[\frac{d}{dz}\left\{\frac{1}{J_0^2(z)}\right\}\right]_{z=0}=0;$$

hence

$$\frac{1}{J_0^2(x)}=1+\sum_{-\infty}^{\infty}\frac{1}{J_1^2(\kappa)}\left\{\frac{1}{(x-\kappa)^2}+\frac{1}{\kappa\,(x-\kappa)}-\frac{x}{\kappa^3}\right\}$$

$$=1+4\sum_{q=1}^{\infty}\frac{1}{J_1^2(\kappa_q)}\left\{\frac{\kappa_q^2}{(x^2-\kappa_q^2)^2}+\frac{1}{x^2-\kappa_q^2}\right\}.$$

But
$$\frac{1}{J_0^2(x)}=1+\frac{x^2}{2}+\frac{5}{32}x^4+\frac{23}{9.64}x^6+\ldots;$$

and therefore

$$\sum_{q=1}^{\infty}\frac{1}{\kappa_q^4 J_1^2(\kappa_q)}=\frac{1}{8},$$

$$\sum_{q=1}^{\infty}\frac{1}{\kappa_q^6 J_1^2(\kappa_q)}=\frac{5}{256},$$

$$\sum_{q=1}^{\infty}\frac{1}{\kappa_q^8 J_1^2(\kappa_q)}=\frac{23}{27.256},$$

while, generally, the value of

$$m \sum_{q=1}^{\infty} \frac{1}{\kappa_q^{2m+2} J_1^2(\kappa_q)}$$

is the coefficient of x^{2m} in the expansion of

$$\left\{1 - \left(\frac{x}{2}\right)^2 \frac{1}{(1\,!)^2} + \left(\frac{x}{2}\right)^4 \frac{1}{(2\,!)^2} - \left(\frac{x}{2}\right)^6 \frac{1}{(3\,!)^2} + \dots\right\}^{-2}$$

in ascending powers of x.

9. Similarly,

$$\frac{1}{J_0^3(x)} = 1 - \sum_{-\infty}^{\infty} \frac{1}{J_1^3} \left\{\frac{1}{(x-\kappa)^3} + \frac{3}{2\kappa} \frac{1}{(x-\kappa)^2} + \frac{1}{2}\left(\frac{1}{1+\kappa^2}\right) \frac{1}{x-\kappa} + \frac{1}{2\kappa} + \frac{x}{4}\left(\frac{1}{\kappa^2} + \frac{1}{\kappa^4}\right)\right\}$$

$$= 1 - 2\sum_{1}^{\infty} \frac{1}{J_1^3(\kappa_q)} \left\{\frac{4\kappa_q^3}{(x^2-\kappa_q^2)^3} + \frac{6\kappa_q}{(x^2-\kappa_q^2)^2} + \left(\frac{\kappa_q}{2} + \frac{2}{\kappa_q}\right) \frac{1}{x^2-\kappa_q^2} + \frac{1}{2\kappa_q}\right\},$$

and

$$\sum_{q=1}^{\infty} \frac{1}{J_1^3(\kappa_q)} \left(\frac{1}{\kappa_q^3} + \frac{4}{\kappa_q^5}\right) = \frac{3}{4},$$

$$\sum_{q=1}^{\infty} \frac{1}{J_1^3(\kappa_q)} \left(\frac{1}{\kappa_q^5} + \frac{16}{\kappa_q^7}\right) = \frac{21}{64},$$

$$\sum_{q=1}^{\infty} \frac{1}{J_1^3(\kappa_q)} \left(\frac{1}{\kappa_q^7} + \frac{36}{\kappa_q^9}\right) = \frac{85}{3.256};$$

while the value of

$$\sum_{q=1}^{\infty} \frac{1}{J_1^3(\kappa_q)} \left(\frac{1}{\kappa_q^{2m+1}} + \frac{4m^2}{\kappa_q^{2m+3}}\right)$$

is the coefficient of x^{2m} in the expansion of

$$\left\{1 - \left(\frac{x}{2}\right)^2 \frac{1}{(1\,!)^2} + \left(\frac{x}{2}\right)^4 \frac{1}{(2\,!)^2} - \left(\frac{x}{2}\right)^6 \frac{1}{(3\,!)^2} + \dots\right\}^{-3}$$

in ascending powers of x.

10. Proceeding to the consideration of the inverse powers of the Bessel functions of positive integral order, all of which are uniform meromorphic functions, we cannot immediately apply Cauchy's theorem to

$$f(z) = \frac{1}{J_n(z)},$$

for it is not regular in the vicinity of the origin. On the other hand, the function

$$\phi(z) = \frac{z^n}{J_n(z)}$$

is regular in the vicinity of the origin; but the integer p, for the contour that has been taken, is $n+1$; and for the expression of the theorem we should require $\phi(0)$, $\phi'(0)$, ..., $\phi^{(n)}(0)$, which are not readily obtainable for a general value of n.

We therefore begin with the function J_1. It has a simple zero at $z=0$. Denoting its other roots (all of which are real) by $\pm\mu_1$, $\pm\mu_2$, $\pm\mu_3$, ..., we have

$$\mu_s = r\pi + \frac{c_1}{8r\pi} - \frac{1}{3\,!}\,\frac{1}{(8r\pi)^3}\,(c_3 - 3c_1c_2 + 2c_1^{\,3} + 48c_1) + \ldots,$$

where

$$\begin{aligned} r &= s + \tfrac{1}{4}, \\ c_1 &= 1 - 4, \\ c_2 &= (1-4)(9-4), \\ c_3 &= (1-4)(9-4)(25-4), \end{aligned}$$

so that $\Sigma \dfrac{1}{\mu_s}$ diverges while $\Sigma \dfrac{1}{\mu_s^{\,2}}$ converges. Thus the function is of class unity.

Owing to the relation between the roots of J_0 and J_1, the quantity $J_0(\mu_s)$ is negative if s is odd, and is positive if s is even; also

$$|J_0(\mu_1)| > |J_0(\mu_2)| > |J_0(\mu_3)| > \ldots .$$

Further, having regard to the asymptotic value of $|J_1(z)|$ for very large values of z, we note that

$$\frac{1}{|zJ_1(z)|}, \quad \frac{1}{|z^2J_1^{\,2}(z)|}, \quad \frac{1}{|z^2J_1^{\,3}(z)|}$$

all tend uniformly to zero for values of z along the large contour that has been selected. Thus, for $1/J_1(z)$, the integer p is equal to unity; for $1/J_1^{\,2}(z)$ and $1/J_1^{\,3}(z)$ it is equal to two.

We need the expansion of $J_1(z)$ in the immediate vicinity of any root μ. Taking $z = \mu + \theta$, we have

$$J_1(z) = \theta J_1' + \frac{\theta^2}{2} J_1'' + \frac{\theta^3}{6} J_1''' + \frac{\theta^4}{24} J_1'''' + \ldots,$$

where the coefficients are the values of the derivatives at μ. Now

$$\begin{aligned} z^2J'' + zJ' + (z^2-1)J &= 0, \\ z^2J''' + 3zJ'' + z^2J' + 2zJ &= 0, \\ z^2J'''' + 5zJ''' + (3+z^2)J'' + 4zJ' + 2J &= 0. \end{aligned}$$

For all values of z, we have

$$2J_n' = J_{n-1} - J_{n+1}, \quad \frac{2n}{z} J_n = J_{n-1} + J_{n+1};$$

hence, for any root μ of J_1, we have

$$2J_1' = J_0 - J_2, \quad 0 = J_0 + J_2,$$

and therefore $\quad J_1'(\mu) = J_0(\mu) = -J_2(\mu)$.

It follows that

$$J_1''(\mu) = -\frac{1}{\mu} J_0(\mu),$$

$$J_1'''(\mu) = -\left(1 - \frac{3}{\mu^2}\right) J_0(\mu),$$

$$J_1''''(\mu) = \left(\frac{2}{\mu} - \frac{12}{\mu^3}\right) J_0(\mu),$$

and so on. Thus

$$J_1(\mu+\theta) = \theta J_0 \left\{1 - \frac{\theta}{2\mu} - \frac{\theta^2}{6}\left(1 - \frac{3}{\mu^2}\right) + \frac{\theta^3}{12}\left(\frac{1}{\mu} - \frac{6}{\mu^3}\right) - \dots\right\}.$$

Consequently, in the vicinity of any root μ of J_1,

$$\frac{1}{J_1(z)} = \frac{1}{\theta J_0}\left\{1 + \frac{\theta}{2\mu} + \frac{\theta^2}{6}\left(1 - \frac{3}{2\mu^2}\right) + \frac{\theta^3}{12}\left(\frac{1}{\mu} + \frac{3}{2\mu^3}\right) + \dots\right\},$$

$$\frac{1}{J_1^2(z)} = \frac{1}{\theta^2 J_0^2}\left\{1 + \frac{\theta}{\mu} + \theta^2\left(\frac{1}{3} - \frac{1}{4\mu^2}\right) + \frac{\theta^3}{3\mu} + \dots\right\},$$

$$\frac{1}{J_1^3(z)} = \frac{1}{\theta^3 J_0^3}\left\{1 + \frac{3\mu}{3\theta} + \tfrac{1}{2}\theta^2 + \theta^3\left(\frac{3}{4\mu} - \frac{1}{4\mu^3}\right)\right\}.$$

Again, from the expression for $J_1(z)$ in powers of z, viz.

$$J_1(z) = \frac{z}{2}\left(\frac{z^2}{8} + \frac{z^4}{3.64} - \frac{z^6}{9.1024} + \frac{z^8}{5.9.2^{14}} - \dots\right),$$

we have

$$\frac{z}{2J_1} = 1 + \frac{z^2}{8} + \frac{z^4}{96} + \frac{7}{9.1024} z^6 + \dots,$$

$$\frac{z^2}{4J_1^2} = 1 + \frac{z^2}{4} + \frac{7}{192} z^4 + \frac{19}{9.512} z^6 + \frac{73}{5.9.4096} z^8 + \dots,$$

$$\frac{z^3}{8J_1^3} = 1 + \frac{3}{8} z^2 + \frac{5}{64} z^4 + \frac{37}{3.1024} z^6 + \frac{379}{3.5.16384} z^8 + \dots .$$

Now consider the function $\dfrac{1}{J_1}$. Manifestly $\dfrac{1}{J_1}-\dfrac{2}{z}$ is regular at the origin, and its value at the origin is zero. The function G for the polar part in the immediate vicinity of any root μ is given by

$$G\left(\frac{1}{z-\mu}\right)=\frac{1}{J_0(\mu)}\,\frac{1}{z-\mu}.$$

The integer p for the function $\dfrac{1}{J_1}-\dfrac{2}{z}$, owing to the contour that has been taken, is unity; so we require the residue of

$$\frac{1}{z}\left(\frac{1}{J_1}-\frac{2}{z}\right)$$

for the root μ. This residue is $\dfrac{1}{\mu}\,\dfrac{1}{J_0(\mu)}$. Consequently, by Cauchy's theorem,

$$\frac{1}{J_1(z)}-\frac{2}{z}=\sum_{-\infty}^{\infty}\frac{1}{J_0(\mu)}\left(\frac{1}{z-\mu}+\frac{1}{\mu}\right)$$

or

$$\frac{1}{J_1(z)}=\frac{2}{z}+2\sum_{r=1}^{\infty}\left\{\frac{z}{z^2-\mu_r^{\,2}}\,\frac{1}{J_0(\mu_r)}\right\},$$

as an expression for $1/J_1(z)$ in rational fractions. When this result is compared with the above, we infer

$$\sum_{r=1}^{\infty}\frac{1}{\mu_r^{\,2}J_0(\mu_r)}=-\frac{1}{8},$$

$$\sum_{r=1}^{\infty}\frac{1}{\mu_r^{\,4}J_0(\mu_r)}=-\frac{1}{96},$$

$$\sum_{r=1}^{\infty}\frac{1}{\mu_r^{\,6}J_0(\mu_r)}=-\frac{7}{9.1024},$$

and so on.

11. We proceed in the same way with the function $1/J_1^{\,2}$. The function

$$\frac{1}{J_1^{\,2}}-\frac{4}{z^2}$$

is regular in the vicinity of the origin; and there its value is equal to unity, while its first derivative vanishes. The function G for the polar part in the immediate vicinity of any root μ is given by

$$G\left(\frac{1}{z-\mu}\right)=\frac{1}{J_0^{\,2}(\mu)}\left\{\frac{1}{(z-\mu)^2}+\frac{1}{\mu(z-\mu)}\right\}.$$

The residue of
$$\frac{1}{z}\left\{\frac{1}{J_1^2(z)}-\frac{4}{z^2}\right\}$$
for a root μ is zero, while the residue of
$$\frac{1}{z^3}\left\{\frac{1}{J_1^2(z)}-\frac{4}{z^2}\right\}$$
is
$$-\frac{1}{\mu^3}\frac{1}{J_0^2(\mu)}.$$
Hence we have
$$\frac{1}{J_1^2(z)}-\frac{4}{z^2}=1+\sum_{-\infty}^{\infty}\frac{1}{J_0^2}\left\{\frac{1}{(z-\mu)^2}+\frac{1}{\mu(z-\mu)}-\frac{z}{\mu^3}\right\}$$
or
$$\frac{1}{J_1^2(z)}=\frac{4}{z^2}+1+4\sum_{r=1}^{\infty}\frac{1}{J_0^2(\mu_r)}\left\{\frac{\mu_r^2}{(z^2-\mu_r^2)^2}+\frac{1}{z^2-\mu_r^2}\right\}.$$
But also
$$\frac{1}{J_1^2(z)}=\frac{4}{z^2}+1+\frac{7}{48}z^2+\frac{19}{9.128}z^4+\frac{73}{5.9.1024}z^6+\dots;$$
and therefore
$$\sum_{r=1}^{\infty}\frac{1}{\mu_r^4 J_0^2(\mu_r)}=\frac{7}{192},$$
$$\sum_{r=1}^{\infty}\frac{1}{\mu_r^6 J_0^2(\mu_r)}=\frac{19}{9.1024},$$
$$\sum_{r=1}^{\infty}\frac{1}{\mu_r^6 J_0^2(\mu_r)}=\frac{73}{5.27.4096},$$
and so on.

In connection with the function $J_1^{-3}(z)$, we note that
$$\frac{1}{J_1^3(z)}-\frac{8}{z^3}-\frac{3}{z}$$
is regular at the origin and vanishes there, while its first derivative is $\frac{5}{8}$. The function G for the polar part in the immediate vicinity of any root μ is given by
$$G\left(\frac{1}{z-\mu}\right)=\frac{1}{J_0(\mu)}\left\{\frac{1}{(z-\mu)^3}+\frac{3}{2\mu(z-\mu)^2}+\frac{1}{2}\frac{1}{z-\mu}\right\}.$$
The residue of
$$\frac{1}{z}\frac{1}{J_1^3(z)}$$
at μ is
$$\left(\frac{1}{2\mu}-\frac{1}{2\mu^3}\right)\frac{1}{J_0^3(\mu)},$$

and the residue of
$$\frac{1}{z^2}\frac{1}{J_1^{\,3}(z)}$$
at μ is
$$\frac{1}{2\mu^2}\frac{1}{J_0^{\,3}(\mu)}.$$
Hence, by Cauchy's theorem,
$$\frac{1}{J_1^{\,3}(x)}-\frac{8}{x^3}-\frac{3}{x}$$
$$=\tfrac{5}{8}x+\sum_{-\infty}^{\infty}\frac{1}{J_0^{\,3}(\mu)}\left\{\frac{1}{(x-\mu)^3}+\frac{3}{2\mu(x-\mu)^2}+\tfrac{1}{2}\frac{1}{x-\mu}\right.$$
$$\left.+\left(\frac{1}{2\mu}-\frac{1}{2\mu^3}\right)+\frac{x}{2\mu^2}\right\}$$
$$=\tfrac{5}{8}x+x\sum_{r=1}^{\infty}\frac{1}{J_0^{\,3}(\mu_r)}\left\{\frac{8\mu_r^{\,2}}{(x^2-\mu_r^{\,2})^3}+\frac{8}{(x^2-\mu_r^{\,2})^2}+\frac{1}{x^2-\mu_r^{\,2}}+\frac{1}{\mu_r^{\,2}}\right\}.$$
But, from the other expansion,
$$\frac{1}{J_1^{\,3}(x)}=\frac{8}{x^3}+\frac{3}{x}+\tfrac{5}{8}x+\frac{37}{3.128}x^3+\frac{379}{15.2048}x^5+\ldots;$$
and therefore
$$\sum_{r=1}^{\infty}\frac{1}{\mu_r^{\,6}}\frac{1}{J_0^{\,3}(\mu_r)}=-\frac{37}{27.128},$$
$$\sum_{r=1}^{\infty}\frac{1}{\mu_r^{\,8}}\frac{1}{J_0^{\,3}(\mu_r)}=-\frac{379}{3.125.2048},$$
and so on.

12. The corresponding results are manifestly obtainable for $J_2^{-1}, J_2^{-2}, \ldots, J_3^{-1}, J_3^{-2}, \ldots$. But the cases must be treated in separate succession, until the full expansions for $J_n^{-1}, J_n^{-2}, \ldots$ are known.

As already indicated, corresponding results can be obtained for Bessel functions of positive non-integer order. It is not necessary to deal with $J_{\frac{1}{2}}(z)$, for its value is $\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\sin z$, and the expressions of inverse powers of $\sin z$ as sums of rational meromorphic functions are known. We shall merely deal with $J_{\frac{3}{2}}(z)$, the value of which is
$$\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left(\frac{\sin z}{z}-\cos z\right),$$
and the positive roots of which (§ 6) are
$$\sigma_q=l\pi-\frac{1}{l\pi}+\frac{11}{24}\frac{1}{(l\pi)^3}-\ldots,$$

where $l = q + \frac{1}{2}$, and $q = 1, 2, \ldots$; and we take the function

$$\frac{1}{z^{\frac{1}{2}} J_{\frac{3}{2}}(z)},$$

which is a uniform function of z. Near the origin

$$J_{\frac{3}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \left(\frac{z^2}{3} - \frac{z^4}{30} + \ldots\right) = \tfrac{1}{3}\left(\frac{2z^3}{\pi}\right)^{\frac{1}{2}} \left(1 - \frac{z^2}{10} + \ldots\right),$$

and therefore, in that vicinity,

$$\frac{1}{z^{\frac{1}{2}} J_{\frac{3}{2}}(z)} = 3\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \left(\frac{1}{z^2} + \tfrac{1}{10} + \text{positive even powers}\right).$$

Consequently,

$$\frac{1}{z^{\frac{1}{2}} J_{\frac{3}{2}}(z)} - 3\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{z^2}, \quad = F(z),$$

is a uniform function of z, which is regular in the vicinity of the origin; and at the origin we have

$$F(0) = \tfrac{3}{10}\left(\tfrac{1}{2}\pi\right)^{\frac{1}{2}}, \quad F'(0) = 0.$$

Also, when (§ 6) we take a contour in the form of a square, centre the origin, sides parallel to the axes in the z-plane, and one side passing through

$$x = m\pi + \tfrac{3}{4}\pi, \quad y = 0,$$

then
$$\left| \frac{1}{z} \left\{ \frac{1}{z^{\frac{1}{2}} J_{\frac{3}{2}}(z)} - 3\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{1}{z^2} \right\} \right|$$

tends uniformly to zero, when the integer $m \to \infty$, all along the contour. Thus the integer $p = 1$; and Cauchy's theorem can be applied.

We need the expansion of $J_{\frac{3}{2}}(z)$ in the immediate vicinity of any root σ. We have, for sufficiently small values of θ,

$$J_{\frac{3}{2}}(\sigma + \theta) = \theta J_{\frac{3}{2}}' + \frac{\theta^2}{2!} J_{\frac{3}{2}}'' + \frac{\theta^3}{3!} J_{\frac{3}{2}}''' + \ldots.$$

Now
$$z^2 J_{\frac{3}{2}}'' + z J_{\frac{3}{2}}' + (z^2 - \tfrac{9}{4}) J_{\frac{3}{2}} = 0,$$

so that, for any root σ,

$$J_{\frac{3}{2}}'' = -1/\sigma\, J_{\frac{3}{2}}'.$$

But
$$2J_{\frac{3}{2}}' = J_{\frac{1}{2}} - J_{\frac{5}{2}}, \quad 0 = J_{\frac{1}{2}} + J_{\frac{5}{2}};$$

and therefore
$$J_{\frac{3}{2}}' = J_{\frac{1}{2}}(\sigma) = \left(\frac{2}{\pi\sigma}\right)^{\frac{1}{2}} \sin\sigma.$$

Hence $$J_{\frac{3}{2}}''(\sigma) = -\left(\frac{2}{\pi\sigma}\right)^{\frac{1}{2}} \frac{\sin\sigma}{\sigma};$$

and, similarly,

$$J_{\frac{3}{2}}'''(\sigma) = -\left(\frac{2}{\pi\sigma}\right)^{\frac{1}{2}} \left(1 - \frac{17}{4\sigma^2}\right) \sin\sigma.$$

Thus, for $z = \sigma + \theta$,

$$J_{\frac{3}{2}}(z) = \theta J_{\frac{1}{2}}(\sigma)\left\{1 - \frac{\theta}{2\sigma} - \frac{\theta^2}{6}\left(1 - \frac{17}{4\sigma^2}\right) - \ldots\right\};$$

and consequently

$$\frac{1}{z^{\frac{1}{2}}J_{\frac{3}{2}}(z)} = \frac{1}{\sigma^{\frac{1}{2}}J_{\frac{1}{2}}(\sigma)}\,(1/\theta + \text{positive powers of } \theta).$$

Thus $$G\left(\frac{1}{z-\sigma}\right) = \frac{1}{\sigma^{\frac{1}{2}}J_{\frac{1}{2}}(\sigma)}\,\frac{1}{z-\sigma}.$$

The residue at σ of $$\frac{1}{z}\cdot\frac{1}{z^{\frac{1}{2}}J_{\frac{3}{2}}(z)}$$

is at once seen to be $$\frac{1}{\sigma}\cdot\frac{1}{\sigma^{\frac{1}{2}}J_{\frac{1}{2}}(\sigma)}.$$

Consequently, by Cauchy's theorem, we have

$$\frac{1}{x^{\frac{1}{2}}J_{\frac{3}{2}}(x)} - 3\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{1}{x^2} = \tfrac{3}{10}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} + \sum_{-\infty}^{\infty}\frac{1}{\sigma_q^{\frac{1}{2}}J_{\frac{1}{2}}(\sigma_q)}\left(\frac{1}{x-\sigma_q} + \frac{1}{\sigma_q}\right)$$

$$= \tfrac{3}{10}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} + \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\sum_{-\infty}^{\infty}\frac{1}{\sin\sigma_q}\left(\frac{1}{x-\sigma_q} + \frac{1}{\sigma_q}\right).$$

As σ_r is very nearly equal to $(q+\frac{1}{2})\pi$, the series on the right is a converging series except for a value of x equal to any one of the quantities σ. The values of σ for the whole range can be combined in equal and opposite pairs; so we add the two terms corresponding to $\pm\sigma_q$, and their sum is

$$\frac{1}{\sin\sigma_q}\left\{\left(\frac{1}{x-\sigma_q} + \frac{1}{\sigma_q}\right) - \left(\frac{1}{x+\sigma_q} - \frac{1}{\sigma_q}\right)\right\} = \frac{2\sigma_q}{\sin\sigma_q}\left(\frac{1}{z^2-\sigma_q^2} + \frac{1}{\sigma_q^2}\right).$$

Thus, finally,

$$\left(\frac{x}{2}\right)^{\frac{3}{2}}\frac{1}{\Pi(\frac{3}{2})}\,\frac{1}{J_{\frac{3}{2}}(x)} = 1 + \tfrac{1}{10}x^2 + \frac{2x^2}{3}\sum_{q=1}^{\infty}\frac{\sigma_q}{\sin\sigma_q}\left(\frac{1}{x^2-\sigma_q^2} + \frac{1}{\sigma_q^2}\right).$$

We at once infer the result

$$\sum_{q=1}^{\infty}\frac{1}{\sigma_q^3\sin\sigma_q} = -\frac{27}{2800},$$

and others of a like character, from comparing expansions for small values of x.

It is, of course, only the fact, that the order of the Bessel function selected is half an odd integer, which introduces the sine-function into the sum on the right-hand side. Had we taken a fractional positive order n, and denoted the corresponding roots by τ_ϱ, we should have had $-J_{n-1}(\tau_\varrho)$ instead of $J_{\frac{1}{2}}(\sigma)$.

Manifestly, any expansion of positive or negative powers of functions J for specific positive orders can be obtained; for the general positive order n, the expansion of $\dfrac{1}{J_n(z)}$, so far as to give the whole of the negative powers, is required.

NOTE ON THE TRANSFORMATIONS OF THE SYLOW SUBGROUPS.

By *G. A. Miller.*

Let $H_1, H_2, \ldots, H_\lambda$ represent the Sylow subgroups of order p^m contained in a group G, and suppose that $\lambda > 1$. The question considered in the present note is how these subgroups are transformed by the subgroups themselves. It is well known that all these subgroups are transformed transitively under G, and that λ is of the form $1 + kp$, but these subgroups are not necessarily transformed transitively under themselves as results directly from the fact that in the icosahedral group the ten subgroups of order 3 transform each of these subgroups into only seven of these ten Sylow subgroups.

To simplify the considerations which follow it is desirable to recall the fact that if any operator of G, whose order is a power of p, transforms into itself one of these λ subgroups it must be contained in this subgroup, as otherwise G would involve a subgroup of order p^{m+1}. Hence it results that no one of these subgroups transforms into itself another of them. That is, each of these subgroups transforms all of them according to a substitution group of degree $\lambda - 1$. Each of the transitive constituents of this substitution group is of degree p^a, since the order of a transitive group is divisible by its degree.

It will now be proved that these λ subgroups transform each of them into $1 + k_1 p$ of the set, where k_1 has the same

value for each of these subgroups. That is, k_1 is an invariant of the set of subgroups of order p^m. To prove the former of these facts it seems convenient to consider the transitive substitution group G' of degree λ according to which the operators of G transform the λ subgroups in question. The subgroup of G', composed of all its substitutions which omit a given letter, has an invariant subgroup of order p^m, and as each of the transitive constituents of the latter subgroup is of degree p^a, each of the transitive constituents of the former must have a degree which is a multiple of p. Let the letter a_1, which is omitted by this subgroup H_1', correspond to H_1.

If a_1 is replaced by another letter a_2 in any substitution of order p^β contained in G', it is also replaced by all the letters which are found in the transitive constituent of H_1', which involves a_2 by some substitution of order p^β contained in G'. That is, a_1 is replaced in the substitutions of G' whose orders are powers of p by either all the letters of a transitive constituent of H_1' or by none of the letters of such a constituent. This proves the fact that H_1 is transformed into exactly $1+k_1p$ of the subgroups of the set $H_1, H_2, \ldots, H_\lambda$ by operators contained in these subgroups, since every substitution, whose order is a power of p, must omit at least one letter of G'.

When $k=1$ then k_1 is evidently also equal to 1. When $k=2$ it is easy to prove that k_1 is also 2. This result is, however, included in the more general theorem that k_1 must exceed unity whenever k exceeds unity. To prove this theorem it may first be noted that if k_1 were unity while k were greater than unity the substitutions of G', whose orders are powers of p, would all be of order p. A cycle of order p in H_1', which involves one letter contained in a cycle of order p involving a_1, can contain only one letter which is not found in the latter cycle. As $k>1$ there is more than one Sylow subgroup in G' in which the cycles which involve a_1 contain the same letters.

Since each Sylow subgroup of G' is of degree kp, it results that the cycle of order p in H_1', having all except one letter in common with the cycles of order p which include a_1, must have one letter in common with some other cycle of order p contained in G'. We have therefore established the following theorem: *If any group of finite order involves more than one Sylow subgroup of order p^m, each of these $1+kp$ Sylow subgroups is transformed into $1+k_1p$ such subgroups by the other Sylow subgroups, where k_1 exceeds unity whenever k exceeds unity.*

THE EVALUATION OF CERTAIN DEFINITE INTEGRALS INVOLVING TRIGONOMETRICAL FUNCTIONS BY MEANS OF FOURIER'S INTEGRAL THEOREM.

By *S. Pollard.*

(A) $\int_0^\infty \frac{\sin y}{y}\,dy.$

1. FOURIER'S integral theorem states that

$$\frac{\pi}{2}\{f(x+0)+f(x-0)\}=\int_0^\infty d\mu\int_{-\infty}^\infty f(t)\cos\mu(x-t)\,dt\ldots(1),$$

provided that

(*a*) $\int_{-\infty}^\infty |f(t)|\,dt$ is finite,

(*b*) $f(t)$ satisfies certain conditions in the immediate neighbourhood of x.

One of these conditions is that $f(t)$ be of bounded variation in the neighbourhood of x. This being so, (1) readily follows from the second mean value theorem and a simplified form of de la Vallée-Poussin's theorem on the change of order of integration in an infinite repeated integral.

A particular kind of function of bounded variation is evidently one which is equal to some constant in a given interval and is zero elsewhere.

Let us take

$$x=0,$$

$$f(t)=1,\quad (0<t<1)$$

$$=0\quad \text{elsewhere.}$$

Fourier's integral theorem is true, and we get

$$\frac{\pi}{2}\{1+0\}=\int_0^\infty d\mu\int_0^1\cos\mu t\,dt=\int_0^\infty\frac{\sin\mu}{\mu}\,d\mu,$$

i.e. $$\int_0^\infty\frac{\sin y}{y}\,dy=\frac{\pi}{2}.$$

2. The question now arises as to whether we cannot obtain the result, still adhering to the same method, but more directly, *i.e.* as to whether we cannot remove the dependence on the general form of Fourier's theorem by giving explicit analysis which is sufficient to prove the theorem for the particular function taken, although it may not be sufficient for functions of bounded variation in general. It will be found that we can do so as follows.

3. LEMMA I.

$$\int_0^{\frac{1}{2}\pi} \frac{\sin m\theta}{\sin\theta}\,d\theta - \int_0^{\frac{1}{2}\pi} \frac{\sin m\theta}{\theta}\,d\theta \to 0 \quad as \quad m \to \infty.$$

The difference between the two integrals is

$$\int_0^{\frac{1}{2}\pi} \sin m\theta \left(\frac{1}{\sin\theta} - \frac{1}{\theta}\right) d\theta$$

$$= \left[-\frac{\cos m\theta}{m}\left(\frac{1}{\sin\theta} - \frac{1}{\theta}\right)\right]_0^{\frac{1}{2}\pi} - \int_0^{\frac{1}{2}\pi} \frac{\cos m\theta}{m}\left(\frac{\cos\theta}{\sin^2\theta} - \frac{1}{\theta^2}\right) d\theta$$

$$= \frac{1}{m} \cdot \cos\frac{m\pi}{2} \cdot \frac{2-\pi}{\pi} + \frac{1}{m}\int_0^{\frac{1}{2}\pi} \cos m\theta \left(\frac{1}{\theta^2} - \frac{\cos\theta}{\sin^2\theta}\right) d\theta.$$

The first term evidently tends to zero. The second term is less in absolute value than

$$\frac{1}{m}\int_0^{\frac{1}{2}\pi} \left(\frac{1}{\theta^2} - \frac{\cos\theta}{\sin^2\theta}\right) d\theta \leq \frac{1}{m}\int_0^{\frac{1}{2}\pi} \left(\frac{1}{\sin^2\theta} - \frac{\cos\theta}{\sin^2\theta}\right) d\theta$$

$$= \frac{1}{m}\int_0^{\frac{1}{2}\pi} \frac{1}{2\cos^2\frac{1}{2}\theta}\,d\theta = \frac{1}{m}\tan\tfrac{1}{4}\pi \to 0.*$$

This completes the lemma.

* The above argument depends, as is easily seen, on the fact that $\frac{1}{\theta^2} - \frac{\cos\theta}{\sin^2\theta}$ is positive or zero in the range $(0, \frac{1}{2}\pi)$. To show this we have

$$\cos\theta \leq 1 - \tfrac{1}{2}\theta^2 + \tfrac{1}{24}\theta^4,$$

$$\sin\theta \geq \theta - \tfrac{1}{6}\theta^3$$

in the range. Thus

$$\frac{\sin^2\theta}{\cos\theta} \geq \frac{(\theta - \frac{1}{6}\theta^3)^2}{1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4} = \theta^2 \cdot \frac{1 - \frac{1}{3}\theta^2 + \frac{1}{36}\theta^4}{1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4}$$

$$\geq \theta^2\{1 + \tfrac{1}{6}\theta^2 - \tfrac{1}{72}\theta^4\} \geq \theta^2,$$

since in the range $\theta < 2$ and $\theta^2 < 4$.

Hence $$\frac{\cos\theta}{\sin^2\theta} \leq \frac{1}{\theta^2}$$

and $$\frac{1}{\theta^2} - \frac{\cos\theta}{\sin^2\theta} \geq 0.$$

LEMMA II. *If m is an odd positive integer, then*

$$\int_0^{\frac{1}{2}\pi} \frac{\sin m\theta}{\sin\theta}\, d\theta = \tfrac{1}{2}\pi.$$

Let $m=2p+1$. It is easily verified that

$$1+2\cos 2\theta + 2\cos 4\theta + \ldots + 2\cos 2p\theta = \frac{\sin(2p+1)\theta}{\sin\theta}.$$

Integrating from 0 to $\frac{1}{2}\pi$, we get at once

$$\tfrac{1}{2}\pi = \int_0^{\frac{1}{2}\pi} \frac{\sin(2p+1)\theta}{\sin\theta}\, d\theta = \int_0^{\frac{1}{2}\pi} \frac{\sin m\theta}{\sin\theta}\, d\theta.$$

This is the lemma.

LEMMA III. $\displaystyle\int_{\frac{1}{2}\pi}^{\infty} \frac{\sin m\theta}{\theta}\, d\theta \to 0$ as $m \to \infty$.

For, by putting $m\theta = \phi$, we get

$$\int_{\frac{1}{2}\pi}^{\infty} \frac{\sin m\theta}{\theta}\, d\theta = \int_{\frac{1}{2}m\pi}^{\infty} \frac{\sin\phi}{\phi}\, d\phi,$$

which tends to zero by the convergence of $\displaystyle\int^{\infty} \frac{\sin m\theta}{\phi}\, d\phi$. Or as follows:

Take $$\phi(m, \chi) = \int_{\frac{1}{2}\pi}^{\chi} \frac{\sin m\theta}{\theta}\, d\theta.$$

If we can show that the double limit of $\phi(m, \chi)$ as m, χ tend simultaneously to infinity is zero, then a well-known theorem gives the lemma at once. For the theorem states that if the double limit as both variables tend together to their limiting values exists, and the single limit as one of the variables tends to its limit exists; then the repeated limit, *i.e.* the limit of this last-mentioned limit as the other variable tends to its limit, exists and is equal to the double limit.

In the case in point the single limit $\lim\limits_{\chi\to\infty} \phi(m, \chi)$ is already known to exist, by the convergence of the integral $\displaystyle\int_{\frac{1}{2}\pi}^{\infty} \frac{\sin m\theta}{\theta}\, d\theta$ for each value of m. Thus, if we can prove that the double limit is zero, we have at once

$$\lim_{m\to\infty} \left[\lim_{\chi\to\infty} \phi(m, \chi)\right] = 0,$$

and this, putting in the value of the inner limit, is the lemma.

To prove that the double limit is zero, we make use of the second mean value theorem. This gives

$$\phi(m, \chi) = \frac{2}{\pi} \int_{\frac{1}{2}\pi}^{\xi} \sin m\theta \, d\theta = \frac{2}{\pi} \left[-\frac{\cos m\theta}{m} \right]_{\frac{1}{2}\pi}^{\xi},$$

where ξ is some number satisfying $\frac{1}{2}\pi \leq \xi \leq \chi$. Thus

$$|\phi(m, \chi)| \leq \frac{4}{\pi m},$$

and it is evident that given any positive number ϵ we can find a number G such that

$$|\phi(m, \chi)| \leq \epsilon,$$

whenever m and x exceed G. This shows that the double limit mentioned is zero and our lemma is proved.*

LEMMA IV. $\displaystyle\int_0^\infty \frac{\sin y}{y} dy = \int_0^\infty \frac{\sin m\theta}{\theta} d\theta$ if $m > 0$.

Put $y = m\theta$ and the result is obtained at once.

THEOREM. $\displaystyle\int_0^\infty \frac{\sin y}{y} dy = \frac{1}{2}\pi.$

By Lemma IV.,

$$\int_0^\infty \frac{\sin y}{y} dy = \int_0^\infty \frac{\sin m\theta}{\theta} d\theta = \lim \int_0^\infty \frac{\sin m\theta}{\theta} d\theta,$$

as m tends to infinity.

By Lemma III.,

$$\lim \int_0^\infty \frac{\sin m\theta}{\theta} d\theta = \lim \int_0^{\frac{1}{2}\pi} \frac{\sin m\theta}{\theta} = \lim \int_0^{\frac{1}{2}\pi} \frac{\sin m\theta}{\sin\theta} d\theta$$

by Lemma I.

Making m take only odd positive integral values we have, by Lemma II.,

$$\lim \int_0^{\frac{1}{2}\pi} \frac{\sin m\theta}{\sin\theta} d\theta = \int_0^{\frac{1}{2}\pi} \frac{\sin m\theta}{\sin\theta} d\theta = \frac{1}{2}\pi.$$

Hence $\displaystyle\int_0^\infty \frac{\sin y}{y} dy = \frac{1}{2}\pi$, and this is the result required.

* This second argument is much longer than the first, but is of a much more general character. It applies to an integral such as $\displaystyle\int_{\frac{1}{2}\pi}^\infty \frac{\sin m\theta}{\theta^2}$, to which the first argument does not apply.

(B). $\int_0^\infty \left(\frac{\sin y}{y}\right)^2 dy.$

SUBSIDIARY THEOREM. *If $\phi(x)$ is integrable in $(0, \xi)$ for all positive values of ξ, and $\phi(x) \to l$ as $x \to \infty$; then*

$$\frac{1}{\xi}\int_0^\xi \phi(x)\,dx \to l \text{ as } x \to \infty.$$

This is the analogue for functions of a continuous variable of Césaro's well-known theorem for functions of a positive integral variable, namely: *If $\phi(n) \to l$ as $n \to \infty$, so does* $\frac{\phi(1)+\phi(2)+\ldots+\phi(n)}{n}$.*

To prove the theorem, we have, since

$$\phi(x) \to l,$$

corresponding to every positive number ϵ, a number X such that

$$|\phi(x) - l| < \epsilon,$$

for $x \geq \mathrm{X}$. Thus, if $\xi > \mathrm{X}$,

$$\frac{1}{\xi}\int_0^\xi \phi(x)\,dx = \frac{1}{\xi}\int_0^{\mathrm{X}} \phi(x)\,dx + \frac{1}{\xi}\int_{\mathrm{X}}^\xi \phi(x)\,dx$$

lies between

$$\frac{1}{\xi}\int_0^{\mathrm{X}} \phi(x)\,dx + \frac{\xi-\mathrm{X}}{\xi}(l-\epsilon) \text{ and } \frac{1}{\xi}\int_0^{\mathrm{X}} \phi(x)\,dx + \frac{\xi-\mathrm{X}}{\xi}(l+\epsilon).$$

Since $\frac{1}{\xi}\int_0^{\mathrm{X}} \phi(x)\,dx \to 0$ and $\frac{\xi-\mathrm{X}}{\xi} \to 1$, this shows that the limits of indetermination of $\frac{1}{\xi}\int_0^\xi \phi(x)\,dx$ lie between $l-\epsilon$ and $l+\epsilon$. ϵ being arbitrary we must have

$$\frac{1}{\xi}\int_0^\xi \phi(x)\,dx \to l.$$

THEOREM. $\int_0^\infty \left(\frac{\sin y}{y}\right)^2 dy = \tfrac{1}{2}\pi.$

From (A) we have

$$\lim_{x\to\infty}\int_0^x \frac{\sin y}{y}\,dy = \tfrac{1}{2}\pi.$$

Writing $\frac{\sin y}{y}$ as $\int_0^1 \cos yt\,dt$, we get

$$\lim_{x\to\infty}\int_0^x dy\int_0^1 \cos yt\,dt = \tfrac{1}{2}\pi.$$

* See Hardy, *A Course of Pure Mathematics*, 2nd ed., p. 160.

Therefore, by the subsidiary theorem,

$$\lim_{\xi\to\infty} \frac{1}{\xi} \int_0^\xi dx \int_0^x dy \int_0^1 \cos yt\,dt = \tfrac{1}{2}\pi \text{.........(2)}.$$

Now for any particular value of ξ, $\cos yt$ is continuous throughout the range

$$0 \leq t \leq 1, \quad 0 \leq y \leq \xi, \quad 0 \leq x \leq \xi,$$

and so we may alter the order of integration, getting

$$\int_0^\xi dx \int_0^x dy \int_0^1 \cos yt\,dt = \int_0^1 dt \int_0^\xi dx \int_0^x \cos yt\,dy$$

$$= \int_0^1 dt \int_0^\xi \frac{\sin tx}{t}\,dx = \int_0^1 \left[-\frac{\cos tx}{t^2}\right]_0^\xi dt = \int_0^1 \frac{1-\cos t\xi}{t^2}\,dt.$$

Putting $t\xi = 2\phi$ this becomes

$$\int_0^{\frac{1}{2}\xi} \frac{1-\cos 2\phi}{4\phi^2} \cdot \xi^2 \cdot \frac{2d\phi}{\xi} = \xi \int_0^{\frac{1}{2}\xi} \left(\frac{\sin\phi}{\phi}\right)^2 d\phi.$$

(2) now gives

$$\lim_{\xi\to\infty} \int_0^{\frac{1}{2}\xi} \left(\frac{\sin\phi}{\phi}\right)^2 d\phi = \tfrac{1}{2}\pi.$$

i.e.

$$\int_0^\infty \left(\frac{\sin\phi}{\phi}\right)^2 d\phi = \tfrac{1}{2}\pi,$$

which is the result required.

It may be noticed that this result is also a particular case of Fourier's integral theorem. We obtain it by considering summability instead of convergence. The standard theorem in this part of the theory is that if $f(t)$ is continuous or has a discontinuity of the first kind only at x, then

$$\int_0^\infty d\mu \int_{-\infty}^\infty f(t) \cos\mu(x-t)\,dt$$

is *summable by arithmetic means* to $\frac{1}{2}\pi\,[f(x+0)+f(x-0)]$, *i.e.*

$$\tfrac{1}{2}\pi\,[f(x+0)+f(x-0)] = \frac{1}{\xi}\int_0^\xi dx' \int_0^{x'} d\mu \int_{-\infty}^\infty f(t)\cos\mu(x-t)\,dt.$$

Our result is obtained by taking $x=0$ and

$$\begin{aligned} f(t) &= 1 \quad (0 \leq t \leq 1) \\ &= 0 \quad \text{elsewhere} \end{aligned}$$

as before. As in (A) the analysis we give is on the lines of that needed to establish the general result, but omits, in virtue of the simplicity of the function taken, several of its complications.

THE PRIMARY ABERRATIONS OF A THIN OPTICAL SYSTEM.

By *T. W. Chaundy*, Christ Church, Oxford.

§ 1. It is convenient to understand by a thin symmetrical optical system one in which the thicknesses and the separations of all the constituent lenses are negligible, and in which therefore all the refracting interfaces are to be regarded as possessing a common vertex. Such a system, although it can, of course, be only approximately realised by concrete embodiments, yet, theoretically considered, is characterized by certain properties which entitle it to be looked on as a distinct category of optical systems, simpler than the general optical system. It is this comparative simplicity which makes it a convenient jumping-off ground in any enquiry into the aberrations of optical systems.

I propose in this paper to restrict attention merely to the "primary" aberrations, by which I mean those given by the second approximation where small quantities up to the second order are retained in lengths measured along the axis, and small quantities to the third order in lengths measured transverse to the axis.

The aberrations are supposed analysed by the line-coordinate method which I have described elsewhere.* Reference will be made to this paper under the symbol C_1. The method relies for its expression of the aberrations on certain optical constants A, B, C, D, E, F, G, H, I, J, K, L, expressions for which are given for the general optical system in a paper read to the *Lond. Math. Soc.* in June, 1918.† Reference will be made to this paper under the symbol C_2. It will be convenient to suppose that this present paper is read in conjunction with a copy of the paper C_2, since very frequent quotation of formulæ, etc., must be made from it.

The results of C_2, although establishing formulæ sufficient for the evaluation of the primary aberration coefficients A, B, ..., K, L, yet present them in a form which a little practical experience shows is very unhandy. The present paper aims to replace them in the case of a thin system by formulæ simpler theoretically and more adapted for actual computation.

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* *Philosophical Magazine*, vol. xxxiv., December, 1917.

† Series 2, vol. xix., part 1, pp 30–50.

§ 2. *Degrees of freedom of a thin system.* For refraction at a single spherical surface from medium μ into medium μ', the vertex of the surface being the origin of both incident and emergent rays, we have

$$\left.\begin{aligned} \mu' m' &= \mu m - (\mu' - \mu)\rho u \\ u' &= u \end{aligned}\right\} \text{ to the first order.}$$

[C_3, § 4, (12), (13), and C_2, § 3, (12)].

Here ρ is the curvature of the spherical surface, m, m' direction cosines of incident and refracted rays, and u, u' their moments about Oz reckoned positive in the sense $y \rightarrow x$.

For a series of refractions at the interfaces of a thin system, which by definition have a common vertex, we take this vertex as common origin and so for a second refraction, again to the first order,

$$\left.\begin{aligned} \mu'' m'' &= \mu' m' - (\mu'' - \mu')\rho' u' \\ u'' &= u' \end{aligned}\right\}.$$

Evidently then at the end of n refractions we should have

$$\left.\begin{aligned} \mu_n m_n &= \mu m - u\Sigma(\mu' - \mu)\rho \\ u_n &= u \end{aligned}\right\}.$$

But in the notation of C_1 and C_2 we should have written

$$\left.\begin{aligned} m_n &= Pm + Qu \\ u_n &= Rm + Su \end{aligned}\right\}.$$

Thus for any thin system

$$\left.\begin{aligned} &P = \mu/\mu_n, \quad Q = -\{\Sigma(\mu' - \mu)\rho\}/\mu_n \\ &R = 0, \quad S = 1 \end{aligned}\right\} \qquad\ldots\ldots\ldots(1).$$

These equations satisfy of course the fundamental identity $PS - QR = \mu/\mu_n$.

For $(\mu' - \mu)\rho$ we may write κ, the power of the surface ρ, giving

$$Q = -(\kappa_1 + \kappa_2 + \ldots + \kappa_n)/\mu_n = -k_n/\mu_n \qquad\ldots\ldots\ldots(2),$$

where k_n* denotes the power of the system comprising the first n surfaces. These are, of course, the standard first-order formulæ of a thin system.

Evidently the systems obtained by stopping short after any $n-k$ of the foregoing refractions are also thin systems, and so characterized by the property $R = 0$. Reference

* The usual capital K is already required among the aberration coefficients.

therefore to the subsidiary systems in terms of which the formulæ of C_2, § 2 (end), are expressed shows that R vanishes for every such subsidiary system.

It follows from the first, second, fourth, and fifth of these formulæ that

$$AR - PG = 0,$$
$$AS - QG = 0,$$
$$CR - PI = 0,$$
$$CS - QI = 0,$$

i.e. since $PS - QR \neq 0$, we must have

$$A = 0 = C = G = I \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Thus for a thin system the fundamental equations take the form

$$m' = \mu m / \mu' + Qu + \tfrac{1}{2}\{Bu(m^2 + n^2) + 2Du(mu + nv) + (Em + Fu)(u^2 + v^2)\},$$

$$u' = u + \tfrac{1}{2}\{Hu(m^2 + n^2) + 2Ju(mu + nv) + (Km + Lu)(u^2 + v^2)\}.$$

These equations have the form

$$\mu' m' - \mu m = m(u^2 + v^2) f(m^2 + n^2,\ mu + nv,\ u^2 + v^2) + u\phi(m^2 + n^2,\ mu + nv,\ u^2 + v^2),$$

$$u' - u = m(u^2 + v^2) g(m^2 + n^2,\ mu + nv,\ u^2 + v^2) + u\psi(m^2 + n^2,\ mu + nv,\ u^2 + v^2),$$

where f, ϕ, g, ψ denote polynomial functional forms.

It is not difficult to prove that to *any* approximation the fundamental equations of a thin system have this form.

Substitution from (1) and (3) in the first three identities of C_2, § 4 (21) gives

$$\left.\begin{aligned} H &= 1 - \mu_0^2/\mu^2 \\ J &= -Q\mu_0/\mu \\ E - QK + \mu_0 L/\mu &= -Q^2\mu_0/\mu \end{aligned}\right\} \ldots\ldots\ldots\ldots(4).$$

Substitution in the identities C_2, § 4 (22) similarly gives

$$\left.\begin{aligned} B &= Q - \mu_0^2\pi/\mu \\ D - E &= -Q\pi\mu_0 \\ I &= \mu_0^2/\mu^2 - 1 \\ J - K &= -Q\mu_0/\mu - \pi\mu_0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(5).$$

Further manipulation of (4) and (5) enables us to write for *ten* of the *twelve* aberration coefficients

$$A=0, \qquad G=0,$$
$$B=Q-\pi\mu_0^2/\mu, \qquad H=1-\mu_0^2/\mu^2,$$
$$C=0, \qquad I=0,$$
$$D=-\mu_0 L/\mu-\mu_0 Q^2/\mu, \qquad J=-\mu_0 Q/\mu,$$
$$E=-\mu_0 L/\mu-\mu_0 Q^2/\mu+Q\pi\mu_0, \quad K=\pi\mu_0.$$

Evidently then the primary aberrations are completely expressible in terms of two of their number, say F and L, together with π, the Petzval sum—a fact of course well known. The primary aberrations of a thin system have thus three degrees of freedom: the first order behaviour of such a system is completely expressible in terms of a single optical constant $\kappa_1+\kappa_2+\ldots+\kappa_n$, the power of the system. If we compare these facts with the corresponding facts for the general system, we remember that the first order behaviour of the general system requires three constants for its expression (say k, $\partial k/\partial\kappa_1$, and $\partial k/\partial\kappa_p$, k being the power of the system, and κ_1, κ_p those of the end surfaces). Its primary aberrations possess, as is known, six degrees of freedom. In these facts lies the characteristic distinction of a thin system to which I have already alluded.

Having established that the primary aberrations of a thin system are always expressible in terms of three optical constants, there still remains for us the choice of what three quantities should ordinarily be employed for this duty, a decision evidently of a practical character. Clearly Petzval's sum, π, should be one of the three. It has the merit, not merely of simplicity, but also of being symmetrical with regard to the two ends of the instrument: that is to say, it is unaltered if the n^{th} medium and the $p-n^{\text{th}}$ be everywhere interchanged. Such a property (which it is convenient to speak of as "reversibility") also belongs to the first order constant $\kappa_1+\kappa_2+\ldots+\kappa_p$, the power of the system.

Mr. T. Smith, in a paper read to the Physical Society,* proposes for his two remaining constants two quantities, of which one is "reversible". I wish to suggest two constants, of which one is reversible, and the other reversible but for a change of sign.

* *Proceedings of the Physical Society*, vol. xxvii., p. 495.

§ 3. *The constant $D+E$.*

One of these two constants is the quantity $D+E$, which I proceed to express in terms of the powers of the constituent surfaces.

From the 10th equation of C_2, § 4 (19 A), we have, since the R of every subsidiary system vanishes, that

$$\mu(ES-QK) = -\mu_0 \Sigma (Q_n - Q_{n-1})(Q_n \mu_n - Q_{n-1}\mu_{n-1})/(\mu_n - \mu_{n-1}).$$

But $\qquad S=1,\ K\mu_0 = \pi = \Sigma \kappa_n / \mu_n \mu_{n-1}$.

Also, from (2), $\qquad Q_n = -(\kappa_1 + \ldots + \kappa_n)/\mu_n$.

Thus, after reduction,

$$\begin{aligned}\mu(2E-\mu_0\pi Q)/\mu_0 &= \Sigma(\kappa_n/\mu_n\mu_{n-1})\{2\mu_n(\kappa_1+\ldots+\kappa_{n-1}) - 2\mu_{n-1}(\kappa_1+\ldots+\kappa_n) \\ &\qquad - (\kappa_1+\ldots+\kappa_p)(\mu_n-\mu_{n-1})\}/(\mu_n-\mu_{n-1}) \\ &= \Sigma(\kappa_n/\mu_n\mu_{n-1})\{(\kappa_1+\ldots+\kappa_{n-1}-\kappa_{n+1}-\ldots-\kappa_p) \\ &\qquad -(\mu_n+\mu_{n-1})\kappa_n/(\mu_n-\mu_{n-1})\}.\end{aligned}$$

It is convenient to write

$$\sigma_n \equiv \kappa_1 + \ldots + \kappa_{n-1} - (\mu_n + \mu_{n-1})\kappa_n/(\mu_n - \mu_{n-1}) - \kappa_{n+1} \ldots - \kappa_p.$$

Also, by (5),

$$2E - \mu_0 \pi Q = D + E \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Thus

$$D + E = (\mu_0/\mu)\Sigma\kappa_n\sigma_n/\mu_n\mu_{n-1} \ \ldots\ldots\ldots\ldots (8).$$

§ 4. *The constant $2(FP-QD)-\mu_0\pi Q$.*

The third quantity which I employ is that just named, $2(FP-QD)-\mu_0\pi Q$. To evaluate it we have that

$$\begin{aligned}(PS-QR)(FP-QD) &= P^2(FS-QL) \\ &\quad - PQ(FR-PL+DS-QJ) + Q^2(DR-PJ),\end{aligned}$$

i.e. by the 7th, 8th, 11th, 12th equations of C_2, § 3 (19 A),

$$\begin{aligned}\mu_0(FP-QD) &= \Sigma S_n(Q_n/\mu_n - Q_{n-1}/\mu_{n-1})\{P(Q_n - Q_{n-1}) \\ &\qquad - Q(P_n - P_{n-1})\}^2 (1/\mu_n - 1/\mu_{n-1})^{-2}.\end{aligned}$$

But $S_n = 1$ and $P_n = \mu_0/\mu_n$ for any thin system.

Thus $\qquad PQ_n - QP_n = \mu_0(\kappa_{n+1}+\ldots+\kappa_p)/\mu_n\mu_p$,

and

$$\begin{aligned}&P(Q_n - Q_{n-1}) - Q(P_n - P_{n-1}) \\ &\quad = (\mu_0/\mu_n\mu_{n-1}\mu_p)\{\mu_{n-1}(\kappa_{n+1}+\ldots+\kappa_p) - \mu_n(\kappa_n+\ldots+\kappa_p)\} \\ &\quad = (\mu_0/\mu_n\mu_{n-1}\mu_p)\{(\mu_{n-1}-\mu_n)(\kappa_{n+1}+\ldots+\kappa_p) - \mu_n\kappa_n\}.\end{aligned}$$

Thus

$$\mu^2 (FP - QD) / \mu_0{}^* = \Sigma\, (Q_n/\mu_n - Q_{n-1}/\mu_{n-1})(\kappa_{n+1} + \ldots + \kappa_p + \mu_n \rho_n)^2$$

on reduction.

Re-arranging the terms under the summation this again becomes

$$\Sigma\, (Q_n/\mu_n) \{(\mu_n\rho_n + \kappa_{n+1} + \ldots + \kappa_p)^2 - (\mu_{n+1}\rho_{n+1} + \kappa_{n+2} + \ldots + \kappa_p)^2\}$$

$$= \Sigma\, (Q_n/\mu_n) \{(\mu_n\rho_n + \kappa_{n+1} + \ldots + \kappa_p)^2 - (\mu_n\rho_{n+1} + \kappa_{n+1} + \ldots + \kappa_p)^2\}$$

$$= \Sigma\, (Q_n/\mu_n)\, \mu_n (\rho_n - \rho_{n+1})\, \{2\,(\kappa_{n+1} + \ldots + \kappa_p) + \mu_n (\rho_n + \rho_{n+1})\}$$

$$= \Sigma\, Q_n\mu_n (\rho_n^2 - \rho^2_{n+1}) + 2\Sigma\, Q_n (\rho_n - \rho_{n+1})\, (\kappa_{n+1} + \ldots + \kappa_p).$$

Again re-grouping the terms under the signs of summation, the foregoing becomes

$$\Sigma \rho_n^2 (\mu_n Q_n - \mu_{n-1} Q_{n-1}) + 2\,\Sigma \rho_n \{ Q_n (\kappa_{n+1} + \ldots + \kappa_p) - Q_{n-1} (\kappa_n + \ldots + \kappa_p)\}$$

$$= -\,\Sigma \rho_n^2 \kappa_n + 2\Sigma\, (\rho_n / \mu_n\mu_{n-1})\, \{\mu_n (\kappa_1 + \ldots + \kappa_{n-1}) (\kappa_n + \ldots + \kappa_p)$$
$$- \mu_{n-1}\, (\kappa_1 + \ldots + \kappa_n)\, (\kappa_{n+1} + \ldots + \kappa_p)\}$$

$$= -\,\Sigma \rho_n^2 \kappa_n + 2\,\Sigma\, (\rho_n/\mu_n\mu_{n-1}) \{(\mu_n - \mu_{n-1}) (\kappa_1 + \ldots + \kappa_n) (\kappa_{n+1} + \ldots + \kappa_p)$$
$$+ \mu_n\kappa_n (\kappa_1 + \ldots + \kappa_{n-1} - \kappa_{n+1} - \ldots - \kappa_p)\}$$

$$= +\,\Sigma \rho_n^2 \kappa_n + 2\Sigma\, (\kappa_n / \mu_n\mu_{n-1})\, \{(\kappa_1 + \ldots + \kappa_n)\, (\kappa_{n+1} + \ldots + \kappa_p)$$
$$+ \mu_n\rho_n\, (\kappa_1 + \ldots + \kappa_{n-1} - \kappa_{n+1} - \ldots - \kappa_p) - \mu_n\mu_{n-1}\rho_n^2\}.$$

Thus $\qquad \{2\,(FP - DQ) - Q^2\mu_0\pi\}\, \mu^2/\mu_0{}^*$

$$= 2\Sigma\, (\mu_n - \mu_{n-1})\, \rho_n^3 + \Sigma\, (\kappa_n/\mu_n\mu_{n-1})\, \{4\, (\kappa_1 + \ldots + \kappa_n)\, (\kappa_{n+1} + \ldots + \kappa_p)$$
$$+ 4\mu_n\rho_n (\kappa_1 + \ldots + \kappa_{n-1} + \kappa_n - \kappa_{n+1} - \ldots - \kappa_p)$$
$$- 4\mu_n^2\rho_n^2 - (\kappa_1 + \ldots + \kappa_p)^2\}$$

$$= 2\Sigma\, (\mu_n - \mu_{n-1})\, \rho_n^3 - \Sigma\, (\kappa_n/\mu_n\mu_{n-1})(\kappa_1 + \ldots + \kappa_n - 2\mu_n\rho_n - \kappa_{n+1} - \ldots - \kappa_p)^2$$

on reduction. But

$$\kappa_1 + \ldots + \kappa_n - 2\mu_n\rho_n - \kappa_{n+1} - \ldots - \kappa_p$$
$$= \kappa_1 + \ldots + \kappa_{n-1} - (\mu_{n-1} + \mu_n)\, \rho_n - \kappa_{n+1} - \ldots - \kappa_p$$

$= \sigma_n$ in the notation of equation (3).

For brevity I further write

$$\left.\begin{aligned} \pi_1 &\equiv \Sigma\, \sigma_n\kappa_n/\mu_n\mu_{n-1} \\ \pi_2 &\equiv \Sigma\, \sigma_n^2\kappa_n/\mu_n\mu_{n-1} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

* μ on the left denotes, of course, the same quantity as μ_p on the right.

the notation indicating the dimension in σ and the analogy with

$$\pi \equiv \Sigma\, \kappa_n / \mu_n \mu_{n-1}.$$

I write also $$h \equiv \Sigma\, (\mu_n - \mu_{n-1})\, \rho_n^3 \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

in analogy with $k \equiv \Sigma\, (\mu_n - \mu_{n-1})\, \rho_n$, the power of the complete system. In this notation we have therefore that

$$\left.\begin{aligned} D+E = 2E - \mu_0 Q\pi = \mu_0 \pi_1 / \mu \\ 2\,(FP - DQ) - \mu_0 Q^2 \pi = 2\mu_0 h / \mu^2 - \mu_0 \pi_2 / \mu^2 \end{aligned}\right\} \ldots\ldots(11).$$

These I propose as the two constants which in conjunction with π should serve as the three fundamental constants in terms of which every primary aberration may be expressed.

Evidently, if we reverse the system, σ_n becomes $-\sigma_{p-n}$; thus π_2 is reversible, while π_1 is reversible with change of sign. By analogy with k it is evident that h is reversible (without change of sign). Hence $2\,(FP - QD) - \mu_0 Q^2 \pi$ is reversible, while $D+E$ is reversible with change of sign (except of course for possible factors μ_0 and μ).

The quantity σ_n may be given an analogy with k_n the power of the first n surfaces, or more precisely with $k_n - k_n'$, where k_n' is the power of the last $p-n$ surfaces, which is possibly useful as a mnemonic or otherwise.

The power of κ of a surface is defined as the curvature of the surface multiplied by the difference of the μ's of the media which it separates. Permit ourselves to define λ, the "power" of a *medium*, as its refractive index multiplied by the difference of the curvatures of the surfaces which it separates, *i.e.* $\lambda = (\rho' - \rho)\, \mu$. The end-media of the system we may suppose to have the plane at infinity as one bounding surface, so that

$$\lambda_0 = -\rho_1 \mu_0\,;\ \ \lambda_p = \rho_p \mu_p.$$

Then

$$\begin{aligned} \lambda_0 + \lambda_1 + \ldots + \lambda_{n-1} &= -\rho_1 \mu_0 + (\rho_1 - \rho_2)\, \mu_1 + \ldots + (\rho_{n-1} - \rho_n)\, \mu_{n-1} \\ &= (\mu_1 - \mu_0)\, \rho_1 + \ldots + (\mu_{n-1} - \mu_n)\, \rho_{n-1} - \mu_{n-1}\rho_n \\ &= \kappa_1 + \kappa_2 + \ldots + \kappa_{n-1} - \mu_{n-1}\rho_n. \end{aligned}$$

So $$\lambda_n + \ldots + \lambda_p = \mu_n \rho_n + \kappa_{n+1} + \ldots + \kappa_p.$$

Hence $$\sigma_n = (\lambda_0 + \ldots + \lambda_{n-1}) - (\lambda_n + \ldots + \lambda_p),$$

which is comparable with

$$k_n - k_n' = (\kappa_1 + \ldots + \kappa_n) - (\kappa_{n+1} + \ldots + \kappa_p).$$

Since $$k = (\lambda_0 + \ldots + \lambda_{n-1}) + (\lambda_n + \ldots + \lambda_p)$$
it is possible to write
$$FP - DQ = \mu_0 h/\mu^2 - (\mu_0/\mu^2)\,\Sigma\,(\lambda_0 + \ldots + \lambda_{n-1}) \times (\lambda_n + \ldots + \lambda_p)\,\kappa_n/\mu_n\mu_{n-1} \ldots\ldots (12).$$
The second term on the right does not however seem so convenient as π_2.

We may now write the twelve primary aberration coefficients of the system as

$$\begin{array}{ll} A = 0, & G = 0, \\ B = -k/\mu - \pi\mu_0^2/\mu, & H = 1 - \mu_0^2/\mu^2, \\ C = 0, & I = 0, \\ D = \mu_0(\pi_1 + \pi k)/2\mu, & J = k\mu_0/\mu^2, \\ E = \mu_0(\pi_1 - \pi k)/2\mu, & K = \pi\mu_0, \\ F = h/\mu - \pi_2/2\mu - k\pi_1/2\mu, & L = -k^2/\mu^2 - (\pi_1 + \pi k)/2. \end{array}$$

From these formulæ the primary aberrations can of course be calculated; thus the primary spherical aberration for an incident parallel beam and semi-aperture y is

$$\tfrac{1}{4}y^2\,\{2\mu h/k^2 - 2k/\mu - \mu\pi + 2\mu\pi_1/k - \mu\pi_2/k^2\}.$$

Other formulæ given in C_1, (7) enable the other aberrations of a beam originally parallel to be stated in terms of h, k, π, π_1, π_2.

In particular, it may be remarked that for a stop at $(\xi, 0, 0)$, *i.e.* a distance ξ *beyond* the first surface, the astigmatism is

$$(\mu/\mu_0)^2\,\{(S\xi - R)\,(F\xi - D) - (Q\xi - P)\,(L\xi - J)\}.$$

If the stop is at the incident focus (*i.e.* the focus in the first medium), $\xi = P/Q$ and the astigmatism reduces to

$$(\mu'/\mu)\,(FP - QD)/Q^2.$$

It is known that this astigmatism is unaltered, except for factors μ, μ_0, by reversal of the optical system. This is corroborated by the result of equation (12) and provides an interpretation of the quantity $FP - QD$. The question may be asked whether a set of six reversible constants (including of course Petzval's sum) can be found for the general system. It is possible to develop a set as follows. The quantity $FP - QD$, it is known, remains reversible for any system

(thick or thin). Operation on this with the differential operators $\partial/\partial\kappa_1 \pm \partial/\partial\kappa_p$ (which are themselves reversible with or without change of sign) evidently generates aberration constants, which are reversible with or without change of sign. In this way, a set of five reversible constants can be added to Petzval's sum to serve for the complete determination of the primary aberrations of the general symmetrical instrument. The constants so determined do not however seem to be in as handy a form as could be desired, and it is possible a more convenient set of reversible constants is discoverable. For this reason I have limited the foregoing discussion to the case of thin systems alone.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

LIV.

Further notes on Mellin's inversion formulæ.

1. The formulæ discussed in Notes XLIX. and LII. lend themselves to various transformations. It is worth while to add a few remarks concerning some of the most interesting formulæ thus obtained.

Suppose, in Theorem A of Note XLIX.*, that $\phi(x) = 0$ when $x > 1$, in which case the second convergence condition may naturally be dropped; and write

$$x = e^{-y}, \quad \phi(x) = \phi(e^{-y}) = \chi(y).$$

* There is a misstatement in the enunciation of this theorem. In lines 6, 7, instead of 'is uniformly convergent throughout any strip $\alpha < \alpha' \leq \sigma \leq \beta' < \beta$, and represents an analytic function $f(s)$ regular in the strip' read 'is uniformly convergent throughout any rectangle

$$\alpha < \alpha' \leq \sigma \leq \beta' < \beta, \quad -T \leq t \leq T,$$

and represents an analytic function $f(s)$ regular in the strip $\alpha < \sigma < \beta$'. The error is verbal only, as the second statement embodies all that is required or used in the proof.

We thus obtain

THEOREM *A*. *Suppose that $\chi(y)$ is integrable (in the sense of Lebesgue) in every interval $0 \leq y \leq \Delta$, and that*

$$\int_0^\infty e^{-\alpha y}\chi(y)\,dy,$$

where α is real, is convergent, so that

$$\int_0^\infty e^{-sy}\chi(y)\,dy \tag{1}$$

is uniformly convergent throughout any region

$$\sigma \geq \alpha' > \alpha, \quad -T \leq t \leq T,$$

and represents an analytic function $f(s)$ regular for $\sigma > \alpha$. Then the integral

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{sy} f(s)\,ds \quad (a > \alpha) \tag{2}$$

is summable $(C, 1)$, to sum $\chi(y)$, for almost all positive values of y. In particular this is true at all points of continuity of $\chi(y)$: more generally, the integral is summable to sum

$$\tfrac{1}{2}\{\chi(y-0)+\chi(y+0)\} \tag{3}$$

whenever this expression has a meaning.

The formulæ

$$f(s) = \int_0^\infty e^{sy}\chi(y)\,dy, \tag{4}$$

$$\chi(y) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{sy} f(s)\,ds \tag{5}$$

are of course very familiar, and have been studied in a variety of forms by different writers*.

It is hardly necessary to point out that (5) embodies a proof of the uniqueness of any solution of the integral equation (4).

2. I add some examples to illustrate the use of the formulæ (4) and (5).

(*a*) Suppose that $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \ldots, \lambda_n \to \infty$, and

$$A(y) = a_1 + a_2 + \ldots + a_n \quad (\lambda_n < y \leq \lambda_{n+1}).$$

* See for example H. Bateman, 'Report on the history and present state of the Theory of Integral Equations', *Reports of the British Association* (Sheffield, 1910), p. 61.

Finally suppose that the series Σa_n is convergent. Then it may be verified at once that

$$(6) \qquad f(s) = \Sigma a_n e^{-\lambda_n s} = \int_0^\infty s e^{-sy} A(y)\, dy$$

if $\sigma > 0$. It follows that

$$(7) \qquad \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{sy}\frac{f(s)}{s}\, ds = a_1 + a_2 + \ldots + a_n$$

if $a > 0$, $\lambda_n < y < \lambda_{n+1}$. If $y = \lambda_{n+1}$ the value of the integral is

$$a_1 + a_2 + \ldots + a_n + \tfrac{1}{2}a_{n+1}.$$

This is the well-known formula of Cahen, Hadamard, and Perron, which is so fundamental in the general theory of Dirichlet's series. It should be observed however that Theorem A shows only that the integral (5) is summable, and not that it is convergent.

It is easily shown that* *if the integral* (1) *is absolutely convergent for* $\sigma > \alpha$, *and* $\chi(y)$ *is of bounded variation in the neighbourhood of the value of* y *considered, then* (2) *is convergent in the sense that*

$$\lim_{T\to\infty}\int_{a-iT}^{a+iT} e^{sy} f(s)\, ds$$

(the principal value of Cauchy) *exists and has the value* (3). Applying this result to the present problem, we obtain Perron's theorem in a form much nearer to that in which it is usually stated†. There is still something lacking; for in fact

$$\lim_{T_1,\, T_2\to\infty}\int_{a-iT_2}^{a+iT_2} e^{sy}\frac{f(s)}{s}\, ds$$

exists, the use of the principal value being only required when y has one of the particular values λ_n. This, however, is of no particular importance; and it is of some interest to have alternative methods for the proof of so fundamental a formula. It is plain that the more general formulæ for the Rieszian means‡

$$\Sigma a_n(\omega - \lambda_n)^\kappa, \quad \Sigma a_n(w - l_n)^\kappa,$$

where $l_n = e^{\lambda_n}$, $w = e^\omega$, may be arrived at in a similar manner.

* See Note XLIX., p. 181.

† See Dr. Riesz's and my tract, 'The general theory of Dirichlet's series', pp. 12–14.

‡ Hardy and Riesz, *l.c.*, pp. 50–51.

3. (*b*) It is easily verified that, if

$$\vartheta(y) = 1 + 2\sum_1^\infty e^{-n^2\pi^2 y},$$

then $$f(s) = \int_0^\infty e^{-sy}\vartheta(y)\,dy = \frac{1}{\sqrt{s}\tanh\sqrt{s}} \quad (\sigma > 0).$$

The simplest method of proof is to substitute for $\vartheta(y)$ from the functional equation

$$\vartheta(y) = \frac{1}{\sqrt{(\pi y)}}\,\vartheta\left(\frac{1}{\pi^2 y}\right)$$

and integrate term by term.

Now the function $f(s)$ satisfies the Riccati's differential equation

$$f^2 + 2f' + \frac{f-1}{s} = 0.$$

Also

$$f^2 = \left(\int_0^\infty e^{-sy}\vartheta(y)\,dy\right)^2 = \int_0^\infty e^{-sy}\,dy\int_0^y \vartheta(u)\,\vartheta(y-u)\,du,$$

$$f' = -\int_0^\infty e^{-sy}\,y\vartheta(y)\,dy,$$

and $$\frac{f}{s} = \frac{1}{s}\int_0^\infty e^{-sy}\vartheta(y)\,dy = \int_0^\infty e^{-sy}\,dy\int_0^y \vartheta(u)\,du.$$

Hence, if we write

$$\phi(y) = \int_0^y \vartheta(u)\,\vartheta(y-u)\,du - 2y\vartheta(y) + \int_0^y \vartheta(y)\,du - 1,$$

we have $$\int_0^\infty e^{-sy}\phi(y)\,dy = f^2 + 2f' + \frac{f-1}{s} = 0,$$

so that $$\phi(y) = 0.$$

The discovery of this very curious integral equation for $\vartheta(y)$ is due to F. Bernstein*. The proof originally given by Bernstein is quite different; but I understand that he had also found the one which I have sketched, and I therefore shall not enter further into its details.

4. (*c*) It is known that†

$$(8) \qquad \int_0^\infty e^{-sy}y^\alpha J_\alpha(my)\,dy = \frac{\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}}\,\frac{(2m)^\alpha}{(m^2+s^2)^{\alpha+\frac{1}{2}}},$$

* F. Bernstein, 'Die Integralgleichung der elliptische Thetanullfunktion', *Berliner Sitzungsberichte*, 21 Oct. 1920. [The proof has been published since this note was written in the *Proceedings of the Royal Society of Amsterdam*.]

† Nielsen, *Cylinderfunktionen*, p. 186.

if $\sigma > 0$, $m > 0$, $\alpha > -\frac{1}{2}$. Hence, if also $\beta > -\frac{1}{2}$, and

$$\phi(y) = \int_0^y u^\alpha (y-u)^\beta J_\alpha(mu) J_\beta(my - mu)\,du,$$

we have

$$\int_0^\infty e^{-sy}\phi(y)\,dy = \frac{\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta+\frac{1}{2})}{\pi}\,\frac{(2m)^{\alpha+\beta}}{(m^2+s^2)^{\alpha+\beta+1}}.$$

Hence we deduce the formula

$$(9)\quad \int_0^y u^\alpha (y-u)^\beta J_\alpha(mu) J_\beta(my-mu)\,du$$
$$= \frac{\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta+\frac{1}{2})}{\Gamma(\alpha+\beta+1)\sqrt{(2\pi m)}}\, y^{\alpha+\beta+\frac{1}{2}} J_{\alpha+\beta+\frac{1}{2}}(my).$$

This formula appears to be new: the particular case in which $\alpha = 0$, $\beta = 0$, viz.

$$(10)\qquad \int_0^y J_0(mu) J_0(my-mu)\,du = \frac{\sin my}{m},$$

has been given before by Kapteyn and Bateman*.

We have also

$$(11)\quad \int_0^\infty e^{-sy} J_\alpha(my)\frac{dy}{y} = \frac{1}{\alpha}\left\{\frac{\sqrt{(m^2+s^2)}-s}{m}\right\}^\alpha$$

if $\alpha > 0$, and

$$(12)\quad \int_0^\infty e^{-sy} J_\alpha(my)\,dy = \frac{1}{\sqrt{(m^2+s^2)}}\left\{\frac{\sqrt{(m^2+s^2)}-s}{m}\right\}^\alpha$$

if $\alpha > -1$. Multiplying two equations of the type (11), or of the types (11) and (12), and pursuing the same line of argument as above, we are led to the formulæ

$$(13)\quad \int_0^y \frac{J_\alpha(mu) J_\beta(my-mu)}{u(y-u)}\,du = \frac{\alpha+\beta}{\alpha\beta}\,\frac{J_{\alpha+\beta}(my)}{y},$$

$$(14)\quad \int_0^y J_\alpha(mu) J_\beta(my-mu)\frac{du}{u} = \frac{1}{\alpha} J_{\alpha+\beta}(my).$$

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* W. Kapteyn, 'On a series of Bessel functions', *Proc. Roy. Soc. Amsterdam*, vol. 8 (1904), pp. 494–500, and 'Recherches sur les fonctions cylindriques', *Mém. Soc. Roy. des Sc. de Liège* (3), vol. 6 (1906), no. 5, pp. 1–24; H. Bateman, 'A generalisation of the Legendre Polynomial', *Proc. London Math. Soc.* (2), vol. 3 (1905), pp. 111–123. I am indebted for these references to Prof. G. N. Watson.

In (13) $\alpha>0$, $\beta>0$; in (14) $\alpha>0$, $\beta>-1$. The latter is given by both Kapteyn and Bateman; and the former is an obvious corollary. The integral

$$(15) \qquad \phi(y)=\int_0^y J_\alpha(mu)\, J_\beta(my-mu)\, du,$$

where $\alpha>-1$, $\beta>-1$, is less simple. Multiplying two equations of the type (12), we obtain

$$\int_0^\infty e^{-sy}\phi(y)\,dy=\frac{1}{m^2+s^2}\left\{\frac{\sqrt{(m^2+s^2)}-s}{m}\right\}^{\alpha+\beta}$$
$$=\frac{2}{m}\left(S^{\alpha+\beta+1}-S^{\alpha+\beta+3}+\ldots\right),$$

where
$$S=\frac{\sqrt{(m^2+s^2)}-s}{m}.$$

And from this we deduce Bateman's formula

$$(16) \qquad \int_0^y J_\alpha(mu)\, J_\beta(my-mu)\, du$$
$$=\frac{2}{m}\left\{J_{\alpha+\beta+1}(my)-J_{\alpha+\beta+3}(my)+\ldots\right\},$$

which also reduces to (10) when $\alpha=0$, $\beta=0$. The series can be summed in finite terms when $\alpha+\beta$ is an integer, by means of the formulæ

$$\cos x=J_0(x)-2J_2(x)+2J_4(x)-\ldots,\quad \sin x=2J_1(x)-2J_3(x)+\ldots.$$

5. There are of course also a variety of formulæ which are, in substance, transformations of those of Note LII. Among these I may quote the following:

$$\left.\begin{aligned} &f(s)=\int_0^\infty x^{s-1}\phi(x)\,dx,\quad g(s)=\int_0^\infty x^{s-1}\psi(x)\,dx,\\ &\int_0^\infty x^{2a-1}\phi(xe^\lambda)\,\psi(xe^{-\lambda})\,dx=\frac{1}{2\pi}\int_{-\infty}^\infty e^{2\lambda it}f(a+it)\,g(a-it)\,dt \end{aligned}\right\},$$

$$\left.\begin{aligned} &f(s)=\int_0^\infty e^{-sy}\phi(y)\,dy,\quad g(s)=\int_0^\infty e^{-sy}\psi(y)\,dy,\\ &\frac{1}{bc}\int_0^y \phi\left(\frac{u}{b}\right)\psi\left(\frac{y-u}{c}\right)du=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{sy}f(bs)\,g(cs)\,ds \end{aligned}\right\},$$

$$\left.\begin{aligned} &f(s)=\int_0^\infty e^{-sy}\phi(y)\,dy,\quad g(s)=\int_0^\infty e^{-sy}\psi(y)\,dy,\\ &\int_\lambda^\infty e^{-2au}\phi(u-\lambda)\,\psi(u+\lambda)\,du=\frac{1}{2\pi}\int_{-\infty}^\infty e^{2\lambda it}f(a+it)g(a-it)\,dt \end{aligned}\right\}.$$

None of these are essentially different from the formulæ of Note LII., and the reader will be able to supply the conditions to be satisfied by the various functions and parameters which occur in them.

If in the last result we take

$$\phi(y) = \psi(y) = y^{\alpha-1},$$

we obtain, after some elementary transformations,

$$\int_0^\infty \frac{\cos 2\lambda t}{(t^2+a^2)^\alpha}\,dt = \frac{\pi e^{-2a\lambda}(2\lambda)^{2\alpha-1}}{\{\Gamma(\alpha)^2\}}\int_0^\infty e^{-4a\lambda t}\{t(1+t)\}^{\alpha-1}\,dt.$$

Each of these integrals is in fact expressible in terms of Bessel's functions*.

ON TRIANGULAR-SYMMETRIC CURVES.

By *Harold Hilton.*

§1. THE name "triangular-symmetric" has been given to curves of the type

$$ax^k + by^k + cz^k = 0 \text{ .... (i)},$$

but their properties have not so far attracted much attention, even though some very familiar curves are capable of projection into this form (*e.g.* $k = \pm 1, \pm\frac{1}{2}, \pm 2, \frac{2}{3}, 3, \ldots$).

We shall suppose k rational. Then the curve is algebraic, and we shall find its Plücker's numbers and discuss its singularities. By

$$n,\ m,\ \delta,\ \kappa,\ \tau,\ \iota,\ D$$

we shall mean respectively the degree, class, number of nodes, number of cusps, number of bitangents, number of inflexions, deficiency of any curve under consideration.

A singularity such as is possessed by

$$a^q y^p = x^{p+q}$$

at the origin, p and q being positive integers, will be called a "branch of type (p, q)". Branches of the types (p, q) and (q, p) are reciprocals of each other. The effect on Plücker's

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* See G. F. Meyer's edition of Dirichlet's lectures on definite integrals, p. 289, and G. H. Hardy, 'Some multiple integrals', *Quarterly Journal*, vol. xxxix. (1908), pp. 357–375 (357–358).

numbers of a branch of type (p, q) is the same as that of $\frac{1}{2}(p-1)(p+q-3)$ nodes, $p-1$ cusps, $\frac{1}{2}(q-1)(p+q-3)$ bitangents, and $q-1$ inflexions.*

By a suitable choice of homogeneous coordinates with the same triangle of reference, the curve (i), supposed real, can be put in the form

$$x^k+y^k=z^k \qquad \text{(ii)},$$

which becomes

$$x^k+y^k=a^k \qquad \text{(iii)}$$

on projecting $z=0$ into the line at infinity.

The condition that this curve should touch $\lambda x+\mu y+1=0$, *i.e.* the tangential equation of (iii), is

$$(-a)^l(\lambda^l+\mu^l)=1 \qquad \text{(iv)},$$

where $$l=k/(k-1).$$

Hence a polar reciprocal of (iii) is a curve of the same kind, but with $k/(k-1)$ replacing k.

If from any point of (iii) we draw perpendiculars to the axes of references, the line joining the feet of these perpendiculars envelopes the curve obtained on replacing k in (iii) by $k/(k+1)$.

Six of the curves obtained by giving $a:b:c$ any values in (i) have three-point contact with a given conic, say

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0.$$

Their six points of osculation lie on a cubic through the intersections of the sides of the triangle of reference with its reciprocal triangle for the given conic, namely, on

$$(ABC+2FGH-AF^2-BG^2-CH^2)\,xyz$$
$$=(k-1)(Ax+Hy+Gz)(Hx+By+Fz)(Gx+Fy+Cz).$$

For algebraic curves, which alone are considered here, $k=\pm p/q$, where p and q are positive integers prime to one another. The case in which $k=-p/q$ can be at once derived from that in which $k=+p/q$ by replacing x, y, a by $1/x$, $1/y$, $1/a$; *i.e.* by a quadratic transformation†. We may therefore consider the two cases together.

The rationalized form of (iii) is

$$\Pi\,(\omega x^k+\omega' y^k-a^k)=0,$$

* See the author's *Plane Algebraic Curves*, p. 65, ex. 4; p. 119, ex. 1.

† See *Plane Algebraic Curves*, p. 120.

the product being taken for all values of ω and ω', such that $\omega^q = \omega'^q = 1$.

Before investigating these curves we shall give in § 2 one or two results which will be useful later.

§ 2. If $f(x, y) = 0$ is an algebraic curve of degree n, $f(x^p, y^p) = 0$ is algebraic of degree np, p being any positive integer.

To each node (a, b) of $f(x, y) = 0$, where a and b are finite and not zero, correspond p^2 nodes of $f(x^p, y^p) = 0$, namely $(a^{1/p}, b^{1/p})$, where any p^{th} roots are taken; and similarly for cusps.

To each tangent $y = \mu x$ to $f(x, y) = 0$ from the origin (not touching at infinity or on $xy = 0$) correspond p tangents $y = \mu^{1/p} x$ from the origin each touching $f(x^p, y^p) = 0$ at p points.

Similarly for tangents parallel to an axis of reference.

Suppose now that $f(x, y) = 0$ cuts $y = 0$ where $x = c$, and that

$$f(x + c,\ y) \equiv \Sigma k x^\alpha y^\beta + \ldots,$$

where $\Sigma k x^\alpha y^\beta = 0$ is any approximation to the shape of $f(x + c, y) = 0$ at the origin, *i.e.* to $f(x, y) = 0$ at $(c, 0)$ as given by Newton's diagram or otherwise. Then the corresponding approximation to $f([x + c^{1/p}]^p, y^p) = 0$ at the origin, *i.e.* to $f(x^p, y^p) = 0$ at $(c^{1/p}, 0)$, is $\Sigma k' x^\alpha y^{\beta p} = 0$, where

$$k' = p^\alpha k c^{\alpha(p-1)/p}.$$

For

$$f([x + c^{1/p}]^p,\ y^p) \equiv \Sigma k \{(x + c^{1/p})^p - c\}^\alpha y^{\beta p} + \ldots = \Sigma k' x^\alpha y^{\beta p} + \ldots.$$

In this k is any constant, and $c^{1/p}$ is any p^{th} root of c, so that to each finite distinct point of $f(x, y) = 0$ on $y = 0$ (other than the origin) correspond p singularities of $f(x^p, y^p) = 0$ on $y = 0$, all of the same kind.

Similarly for $x = 0$ or the line at infinity.

§ 3. We now consider the algebraic curves

$$x^{p/q} + y^{p/q} = a^{p/q} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

where p and q are positive integers.

The case $p = 1$ has been discussed*. It is the general unicursal curve of degree q with the axes of reference and line

* *Messenger of Mathematics*, vol. xlix. (1920), p. 132.

at infinity as tangents of q-point contact. From the results of the paper just quoted, from § 2, and from the tangential equation of § 1 (iv), we readily obtain the following properties.

For the curve (i), when $p > q$,

$$n = pq, \quad m = p(p-q),$$

$$\delta = \{\tfrac{3}{2}p(q-1)(p-3)\} + \{\tfrac{1}{2}p^2(q-1)(q-2)\}$$
$$= \tfrac{1}{2}p(q-1)(pq+p-9),$$

$$\kappa = \{3p(q-1)\} + \{0\} = 3p(q-1),$$

$$\tau = \{\tfrac{3}{2}p(p-q-1)(p-3)\} + \{\tfrac{1}{2}p^2(p-q-1)(p-q-2)\}$$
$$= \tfrac{1}{2}p[p(p-q)^2 - 10p + 9q + 9],$$

$$\iota = \{3p(p-q-1)\} + \{0\} = 3p(p-q-1),$$

$$D = \tfrac{1}{2}(p-1)(p-2)^*.$$

In these expressions for δ, κ, τ, ι the term in the first of the brackets { } refers to singularities at infinity or on the axes of reference. The term in the second of the brackets { } refers to ordinary nodes, etc., situated elsewhere.

Any point on the curve (i) is

$$x = at^{q/p}, \quad y = a(1-t)^{q/p} \ldots\ldots\ldots\ldots\ldots \text{(ii)},$$

and the $\frac{1}{2}p^2(q-1)(q-2)$ ordinary nodes are given by

$$t = \sin r\pi/q \ \operatorname{cosec} (r-s)\pi/q \ e^{\pm s\pi i/q},$$

where r, s are any of the numbers $1, 2, \ldots, q-1$ such that $r > s$.

These nodes are unreal, except for $\frac{1}{2}(q-1)(q-2)$ real acnodes when p is odd and $\frac{1}{2}(q-1)(q-3)$ real acnodes when p is even.

The ordinary bitangents are obtained similarly from the tangential equation of § 1 (iv).

Since the line at infinity is a tangent of q-point contact of $x^{1/q} + y^{1/q} = a^{1/q}$, therefore by § 2 the singularities of (i) at infinity consist of p branches of type $(q, p-q)$, the tangent at each of which passes through the origin. These tangents are the only tangents from the origin to the curve.

There are similarly p such branches on each axis of reference, the tangents being parallel to the other axis; as is at once evident from the symmetry of § 1 (ii).

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* See *Plane Algebraic Curves*, p. 377, ex. 3.

§ 4. For the curve of § 3 (i), when $p<q$,

$$n=pq, \quad m=2p(q-p),$$

$$\delta=\{\tfrac{3}{2}p(p-1)(q-3)\}+\{\tfrac{1}{2}p^2(q-1)(q-2)\}$$
$$=\tfrac{1}{2}p(pq^2-7p-3q+9),$$

$$\kappa=\{3p(p-1)\}+\{0\}=3p(p-1),$$

$$\tau=\{\tfrac{3}{2}p[p(q-p)(q-p+1)+3p-4q+3]\}$$
$$+\{\tfrac{1}{2}p^2(q-p-1)(q-p-2)\}$$
$$=\tfrac{1}{2}p[4p(q-p)^2+11p-12q+9],$$

$$\iota=\{3p(q-p-1)\}+\{0\}=3p(q-p-1),$$

$$D=\tfrac{1}{2}(p-1)(p-2).$$

There are now p branches of type $(p, q-p)$ at infinity, at each of which the line at infinity is the tangent; and similarly for the axes of x and y. The only tangents from the origin are the axes of reference. The ordinary nodes are found as in the case $p>q$.

§ 5. For the curve

$$x^{-p/q}+y^{-p/q}=a^{-p/q}\text{....................(i)},$$

we have $$n=2pq, \quad m=p(p+q),$$

$$\delta=\{\tfrac{3}{2}p(pq^2+pq-p-4q+3)\}+\{\tfrac{1}{2}p^2(q-1)(q-2)\}$$
$$=\tfrac{1}{2}p(4pq^2-p-12q+9),$$

$$\kappa=\{3p(q-1)\}+\{0\}=3p(q-1),$$

$$\tau=\{\tfrac{3}{2}p(p-1)(p+q-3)\}+\{\tfrac{1}{2}p^2(p+q-1)(p+q-2)\}$$
$$=\tfrac{1}{2}p[p(p+q)^2-10p-3q+9],$$

$$\iota=\{3p(p-1)\}+\{0\}=3p(p-1),$$

$$D=\tfrac{1}{2}(p-1)(p-2).$$

At the origin is a multiple point with p branches of type (q, p), the tangents to the branches being $x^p=y^p$. Similarly for the points at infinity on the axes of reference. The double points are obtained by replacing x, y, a by $1/x$, $1/y$, $1/a$ in § 3.

§ 6. If in the curve § 3 (i) we project the points at infinity on the axes of reference into the circular points at infinity,

we obtain a real curve with the same Plücker's numbers and the same kinds of singularity as those of § 3 (i); but some of the singularities which were real in § 3 (i) may be now unreal, and *vice versâ*.

We obtain the polar equation of the projection on replacing x, y in § 3 (i) by $re^{\theta i}$, $re^{-\theta i}$. It is

$$r^{p/q}\cos p\theta/q = b^{p/q} \qquad\text{(i)},$$

where $a/b = 2^{q/p}$. Similarly from § 5 (i) we get

$$r^{p/q} = b^{p/q}\cos p\theta/q \qquad\text{(ii)},$$

where $b/a = 2^{q/p}$.

These curves (i) and (ii) are inverses of each other with respect to the pole. Many of their properties are very familiar. For instance, the angle between any two tangents is proportional to the angle subtended at the pole by their points of contact. Less well known perhaps are their Plücker's numbers and singularities, which are immediately deducible by projection from those of the curves in §§ 3, 4, 5.

The curves (i), (ii) consist of p branches or loops. They have the symmetry of the regular polygon of p sides, and have real crunodes on their axes of symmetry, which are $\frac{1}{2}p(q-1)$ or $\frac{1}{2}p(q-2)$ in number, according as q is odd or even (excluding the pole).

The real foci of (i) all coincide at the pole if $q > p$, and coincide in sets at the points

$$r = 2^{q/p}b, \quad \theta = 2j\pi/p \quad (j = 1, 2, \ldots, p),$$

if $p > q$.

The curve (ii) has no ordinary foci. The real singular foci coincide in sets at

$$r = b/2^{q/p}, \quad \theta = 2j\pi/p \quad (j = 1, 2, \ldots, p).$$

THE BERNOULLIAN FUNCTIONS OCCURRING IN THE ARITHMETICAL APPLICATIONS OF ELLIPTIC FUNCTIONS.

By *E. T. Bell.*

1. AFTER Glaisher's discussion of the Bernoullian function* it may seem superfluous to add anything further on sums of like powers of the natural numbers, as doubtless the formulæ given below either are included in his or may without difficulty be derived from them. Nevertheless, it is desirable to consider a set of sixteen such sums in the special forms, differing from those convenient for most purposes, that are immediately available for use in the arithmetic of elliptic functions. This set, as shown in §§ 12, 13, contains all the Bernoullian functions that can arise in such work; and from the discussion there it is evident that the appearance of these functions among the arithmetical consequences of identities between elliptic or other quotients of theta functions depends upon the infinities of the quotients involved.

2. Henceforth we shall use m, n, r, s to denote integers > 0, of which m is always odd and the rest arbitrary. As customary, we denote the numbers of Bernoulli, Euler, and Genocchi by B, E, G, the notation being that of Lucas,† and write

$$\beta_n = 2^n B_n, \quad \gamma_n = 2^n G_n = 2^{n+1}(1-2^n) B_n, \quad R_n = (1-2^{n-1}) B_n;$$

so that by our convention β, E, γ, R do not occur with suffix zero. This remark is of importance in interpreting the symbolic formulæ. Thus, $\cos\gamma x = 1 - \gamma_2/2!$ etc., and *not* as it would be in the notation of Blissard and Lucas, $\gamma_0 - \gamma_2/2!$ etc. The generators of these numbers in symbolic form are

(1) $$x \cot x = \cos\beta x,$$

(2) $$\sec x = \cos Ex,$$

(3) $$2x \tan x = \cos\gamma x - 1,$$

(4) $$x \operatorname{cosec} x = 2 \cos Rx - 1,$$

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* *Quarterly Journal*, vol. xxix., pp. 1–168, continued in vol. xlii., pp. 86–155; vol. xxx., pp. 166–204; vol. xxxi., pp. 193–227; *Messenger of Mathematics*, vol. xxviii., pp. 36–79. Glaisher points out (first paper, pp. 159–168) that well-known formulæ and the symbolic method usually attributed to Lucas were given by Blissard, who anticipated Lucas (*Quarterly Journal*, vol. iv., pp. 279–305, continued in vol. v.) by 15 years. This method is used in the present paper.

† *Théorie des Nombres*, chap. xiv. The R-numbers are defined, *ibid.*, p. 254. For passing from Lucas' notation to Glaisher's, cf. § 10.

the -1 in the last two entering through our convention. Similarly, we have (symbolically)

$$(\gamma+n)^2=\gamma_2+2\gamma_1 n+n^2, \quad \textit{not} \quad \gamma_2+2\gamma_1 n+\gamma_0 n^2, \quad \text{etc.}$$

3. The sixteen formulæ which we shall consider involve a function defined by

(5) $$\phi_n(p,q)=(p+q)^n+(p-q)^n, \quad \phi_0(p,q)=2;$$

so that, symbolically,

(6) $$2\sin px \cos qx=\sin\phi(p,q)x,$$

(7) $$2\cos px \cos qx=\cos\phi(p,q)x,$$

as is evident on writing the left members as sums. If N is any one of β, E, γ, R, we have therefore

$$\phi_s(N,n)=2\left[N_s+\binom{s}{2}N_{s-2}n^2+\ldots\right],$$

the last term in brackets being n^s or $\binom{s}{1}N_1 n^{s-1}$, according as s is even or odd. But if $f_0(r)$, $f_1(r)$, ... are any functions of r (other than β_r, E_r, γ_r, R_r), we retain the usual procedure of the symbolic method and write

$$\phi_s\{f(r),n\}=2\left[f_s(r)+\binom{s}{2}f_{s-2}(r)n^2+\ldots\right],$$

the last term being $n^s f_0(r)$, when s is even. Thus the symbolic polynomial

$$\{f(r)+n\}^4+\{f(r)-n\}^4=2\left[f_4(r)+6n^2 f_2(r)+n^4 f_0(r)\right].$$

An important complementary function $\psi_n(p,q)$ is defined in § 14.

4. It is convenient to express all sums of like powers of the natural numbers arising in the applications of elliptic functions by means of four special sums as follows, of which, note, the first and fourth are defined only for even values of the argument, and the second and third only for odd:

$$T_s(2n)=2^s+4^s+6^s+\ldots+(2n-2)^s,$$

$$\Theta_s(m)=(-1)^{(m-1)/2}\left[-2^s+4^s-6^s+\ldots+(-1)^{(m-1)/2}(m-1)^s\right],$$

$$O_s(m)=1^s+3^s+5^s+\ldots+(m-2)^s,$$

$$\Omega_s(2n)=(-1)^n\left[-1^s+3^s-5^s+\ldots+(-1)^n(2n-1)^s\right].$$

With these we require the conventional values

$$T'_s(2) = \Theta_s(1) = O_s(1) = 0.$$

From the definitions we have

$$T_s(m+1) = 2^s + 4^s + 6^s + \ldots + (m-1)^s,$$

$$\Theta_s(2n-1) = (-1)^{n-1}[-2^s + 4^s - 6^s + \ldots + (-1)^{n-1}(2n-2)^s],$$

$$O_s(2n+1) = 1^s + 3^s + 5^s + \ldots + (2n-1)^s,$$

$$\Omega_s(m-1) = (-1)^{(m-1)/2}[-1^s + 3^s - 5^s + \ldots + (-1)^{(m-1)/2}(m-2)^s].$$

The sixteen cases of these eight obtained by putting $s=2r$, $s=2r-1$, give all the sums occurring in the applications which we have in view, and moreover these forms are those presenting themselves naturally, so that we can pass to special results such as recurrences for sums of divisors, etc., without further transformations. An illustration is given in § 15.

5. The required formulæ are those in (1.1)–(4.4), which are obtained from (1)–(4) on multiplying each in turn by

$$\sin 2nx, \quad \sin mx, \quad \cos 2nx, \quad \cos mx,$$

applying (6) or (7) to the right-hand members of the identities thus obtained, and finally equating coefficients of like powers of x. We shall take the cases in the order indicated.

6. Proceeding with (1) as outlined, we use the identities

$$\sin 2nx \cot x = 1 + \cos 2nx + 2 \sum_{r=1}^{n-1} \cos 2rx,$$

$$\sin mx \cot x = \cos mx + 2 \sum_{r=1}^{(m-1)/2} \cos(2r-1)x,$$

$$\cos 2nx \cot x = \cot x - \sin 2nx - 2 \sum_{r=1}^{n-1} \sin 2rx,$$

$$\cos mx \cot x = \operatorname{cosec} x - \sin mx - 2 \sum_{r=1}^{(m-1)/2} \sin(2r-1)x$$

to reduce the left members of the identities

$$x \sin 2nx \cot x = \sin 2nx \cos \beta x, \text{ etc.}$$

In the last two we must first multiply throughout by x, replace then $x \cot x$, $x \operatorname{cosec} x$ by their β, R equivalents and

expand these before equating coefficients. We find at once

(1.1) $\phi_{2s+1}(2n, \beta) = 2(2s+1)[(2n)^{2s} + 2T_{2s}(2n)]$,

(1.2) $\phi_{2s+1}(m, \beta) = 2(2s+1)[m^{2s} + 2O_{2s}(m)]$,

(1.3) $\phi_{2s}(2n, \beta) - 2\beta_{2s} = 4s[(2n)^{2s-1} + 2T_{2s-1}(2n)]$,

(1.4) $\phi_{2s}(m, \beta) - 4R_{2s} = 4s[m^{2s-1} + 2O_{2s-1}(m)]$.

7. The set for Euler's numbers comes in the same way from (2) and the identities

$$\sin 2nx \sec x = 2(-1)^n \sum_{r=1}^{n} (-1)^r \sin(2r-1)x,$$

$$\sin mx \sec x = (-1)^{(m-1)/2}\left[\tan x + 2 \sum_{r=1}^{(m-1)/2} (-1)^r \sin 2rx\right],$$

$$\cos 2nx \sec x = (-1)^n\left[\sec x + 2 \sum_{r=1}^{n} (-1)^r \cos(2r-1)x\right],$$

$$\cos mx \sec x = (-1)^{(m-1)/2}\left[1 + 2 \sum_{r=1}^{(m-1)/2} (-1)^r \cos 2rx\right].$$

We find

(2.1) $\phi_{2s-1}(2n, E) = 4\Omega_{2s-1}(2n)$,

(2.2) $\phi_{2s-1}(m, E) = \frac{1}{2s}(-1)^{(m+1)/2}\gamma_{2s} + 4\Theta_{2s-1}(m)$,

(2.3) $\phi_{2s}(2n, E) = 2[(-1)^n E_{2s} + 2\Omega_{2s}(2n)]$,

(2.4) $\phi_{2s}(m, E) = 4\Theta_{2s}(m)$.

8. The sets for γ, R may be easily derived by the symbolic method from those for E, β respectively, but the connection with elliptic functions will be more clearly exhibited by establishing them independently in the same way as the foregoing.

From (3) and the identities

$$\sin 2nx \tan x = (-1)^{n-1}\left[1 + (-1)^n \cos 2nx + 2 \sum_{r=1}^{n-1} (-1)^r \cos 2rx\right],$$

$$\sin mx \tan x = (-1)^{(m-1)/2}\Big[\sec x - (-1)^{(m-1)/2}\cos mx$$
$$+ 2 \sum_{r=1}^{(m-1)/2} (-1)^r \cos(2r-1)x\Big],$$

$$\cos 2nx \tan x = (-1)^n\left[\tan x + (-1)^n \sin 2nx + 2 \sum_{r=1}^{n-1} (-1)^r \sin 2rx\right],$$

$$\cos mx \tan x = \sin mx - 2(-1)^{(m-1)/2} \sum_{r=1}^{(m-1)/2} (-1)^r \sin(2r-1)x,$$

we have the following for the γ-numbers:

$$(3.1)\quad \phi_{2s+1}(2n, \gamma) + 2(-1)^n (2n)^{2s+1}$$
$$= 4(2s+1)[-(2n)^{2s} + 2\Theta_{2s}(2n-1)],$$

$$(3.2)\quad \phi_{2s+1}(m, \gamma) - 2m^{2s+1}$$
$$= 4(2s+1)[(-1)^{(m-1)/2} E_{2s} - m^{2s} + 2\Omega_{2s}(m-1)],$$

$$(3.3)\quad \phi_{2s}(2n, \gamma) - 2(2n)^{2s} - 2(-1)^n \gamma_{2s}$$
$$= 8s[-(2n)^{2s-1} + 2\Theta_{2s-1}(2n-1)],$$

$$(3.4)\quad \phi_{2s}(m, \gamma) - 2m^{2s}$$
$$= 8s[-m^{2s-1} + 2\Omega_{2s-1}(m-1)].$$

9. For the numbers R we use (4) and the identities

$$\sin 2nx \operatorname{cosec} x = 2 \sum_{r=1}^{n} \cos(2r-1)x,$$

$$\sin mx \operatorname{cosec} x = 1 + 2 \sum_{r=1}^{(m-1)/2} \cos 2rx,$$

$$\cos 2nx \operatorname{cosec} x = \operatorname{cosec} x - 2 \sum_{r=1}^{n} \sin(2r-1)x,$$

$$\cos mx \operatorname{cosec} x = \cot x - 2 \sum_{r=1}^{(m-1)/2} \sin 2rx,$$

finding

$$(4.1)\quad \phi_{2s+1}(2n, R) - (2n)^{2s+1} = 2(2s+1) O_{2s}(2n+1),$$

$$(4.2)\quad \phi_{2s+1}(m, R) - m^{2s+1} = 2(2s+1) T_{2s}(m+1),$$

$$(4.3)\quad \phi_{2s}(2n, R) - (2n)^{2s} - 2R_{2s} = 4s O_{2s-1}(2n+1),$$

$$(4.4)\quad \phi_{2s}(m, R) - m^{2s} - \beta_{2s} = 4s T_{2s-1}(m+1).$$

10. For purposes of comparison with Glaisher's results and to provide the means for applying some of his numerous theorems to formulæ of the kind illustrated in § 15, we shall write down the U, V equivalents of the functions in § 4, which we have taken as fundamental. From p. 117 of his first paper we find the following relations between his functions and our T, Θ:

$$(s+1)\, T_s(2n) = 2^s [V_{s+1}(n) - V_{s+1}],$$

$$(s+1)\, \Theta_s(m) = 2^s [(-1)^{(m+1)/2} U_{s+1} - U_{s+1}\{(m+1)/2\}];$$

and from pp. 139, 143 of this second paper these for the O, Ω:

$$(s+1)\, O_s(m) = 2^s\left[V_{s+1}(\tfrac{1}{2}m) - V_{s+1}(\tfrac{1}{2})\right],$$

$$(s+1)\, \Omega_s(2n) = 2^s\left[U_{s+1}(n+\tfrac{1}{2}) + (-1)^{n-1}U_{s+1}(\tfrac{1}{2})\right].$$

Following Lucas we have written $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, etc., and these numbers coincide with the V_s of Glaisher (first paper, p. 116), so that our B_{2s} is his V_{2s}, and $V_{2s-1} = B_{2s-1} = 0$. It may be of interest to remark in passing that (1.1)–(1.4) and (4.1)–(4.4), upon elimination of O, T, give relations between the β, R, and similarly for (2.1)–(2.4) and (3.1)–(3.4) and the γ, E. On further eliminating the powers not contained in the ϕ-functions from the resulting eight relations we find formulæ connecting ϕ-functions alone.

11. Consider the following sums

$$\Sigma_n T_s(2d), \quad \Sigma_n'\Theta_s(\delta), \quad \Sigma_n' O_s(\delta), \quad \Sigma_n \Omega_s(2d),$$

$$\Sigma_n' T_s(\delta+1), \quad \Sigma_n \Theta_s(2d-1), \quad \Sigma_n O_s(2d+1), \quad \Sigma_n'\Omega_s(\delta-1),$$

in which Σ_n refers to all divisors d of n that are of a specified kind (L), and Σ_n' refers to all the odd divisors δ of n that are of a specified kind (M). Let $\lambda_s(n)$, $\mu_s(n)$ denote respectively the sums of the s^{th} powers of all the divisors of n of the kind (L), and of all the odd divisors of n of the kind (M). For example, (L) may specify all divisors whose conjugates are odd; and the odd divisors defined by (M) may be all those divisible by 3. For explicitness we should indicate the (L) with Σ_n and the (M) with Σ_n', but to simplify the writing we have omitted this detail.

Sums of this kind are among the most frequent of those occurring in the applications, and the formulæ (1.1)–(4.4) at once evaluate them in terms of ϕ-functions. We recall the conventions regarding zero suffixes stated in § 3. Clearly $\lambda_0(n)$ is the number of divisors of kind (L), and similarly for $\mu_0(n)$. These sums being of much use we shall write down from (1.1)–(4.4) the complete set of formulæ giving them. The numbering corresponds to that in §§ 6–9; thus (1.11) is from (1.1):

(1.11) $\quad \phi_{2s+1}\{2\lambda(n), \beta\} = 2(2s+1)\left[2^{2s}\lambda_{2s}(n) + 2\Sigma_n T_{2s}(2d)\right],$

(1.21) $\quad \phi_{2s+1}\{\mu(m), \beta\} = 2(2s+1)\left[\mu_{2s}(m) + 2\Sigma'_m O_{2s}(\delta)\right],$

(1.31) $\quad \phi_{2s}\{2\lambda(n), \beta\} - 2\beta_{2s}\lambda_0(n)$

$$= 4s\left[2^{2s-1}\lambda_{2s-1}(n) + 2\Sigma_n T_{2s-1}(2d)\right],$$

(1.41) $\phi_{2s}\ \{\mu\ (m),\ \beta\} - 4R_{2s}\mu_0(m)$
$$= 4s\left[\mu_{2s-1}(m) + 2\Sigma'_m O_{2s-1}(\delta)\right].$$

(2.11) $\phi_{2s-1}\{2\lambda(n),\ E\} = 4\Sigma_n\Omega_{2s-1}(2d),$

(2.21) $\phi_{2s-1}\{\mu\ (m),\ E\} = \dfrac{1}{2s}(-1)^{(m+1)/2}\gamma_{2s}\mu_0(m) + 4\Sigma'_m\Theta_{2s-1}(\delta),$

(2.31) $\phi_{2s}\ \{2\lambda(n),\ E\} = 2\left[(-1)^n E_{2s}\lambda_0(n) + 2\Sigma_n\Omega_{2s}(2d)\right],$

(2.41) $\phi_{2s}\ \{\mu\ (m),\ E\} = 4\Sigma'_m\Theta_{2s}(\delta).$

(3.11) $\phi_{2s+1}\{2\lambda(n),\ \gamma\} + 2^{2s+2}(-1)^n\lambda_{2s+1}(n)$
$$= 4(2s+1)\left[-2^{2s}\lambda_{2s}(n) + 2\Sigma_n\Theta_{2s}(2d-1)\right],$$

(3.21) $\phi_{2s+1}\{\mu(m),\ \gamma\} - 2\mu_{2s+1}(m)$
$$= 4(2s+1)\left[(-1)^{(m-1)/2}E_{2s}\mu_0(m) - \mu_{2s}(m) + 2\Sigma_m{}'\Omega_{2s}(\delta-1)\right],$$

(3.31) $\phi_{2s}\ \{2\lambda(n),\ \gamma\} - 2^{2s+1}\lambda_{2s}(n) - 2(-1)^n\gamma_{2s}\lambda_0(n)$
$$= 8s\left[-2^{2s-1}\lambda_{2s-1}(n) + 2\Sigma_n\Theta_{2s-1}(2d-1)\right],$$

(3.41) $\phi_{2s}\ \{\mu(m),\ \gamma\} - 2\mu_{2s}(m) = 8s\left[-\mu_{2s-1}(m) + 2\Sigma_m\Omega'_{2s-1}(\delta-1)\right],$

(4.11) $\phi_{2s+1}\{2\lambda(n),\ R\} - 2^{2s+1}\lambda_{2s+1}(n) = (2s+1)\Sigma_n O_{2s}(2d+1),$

(4.21) $\phi_{2s+1}\{\mu(m),\ R\} - \mu_{2s+1}(m) = 2(2s+1)\Sigma_m{}'T_{2s}(\delta+1),$

(4.31) $\phi_{2s}\ \{2\lambda(n),\ R\} - 2^{2s}\lambda_{2s}(n) - 2R_{2s}\lambda_0(n)$
$$= 4s\Sigma_n O_{2s-1}(2d+1),$$

(4.41) $\phi_{2s}\ \{\mu(m),\ R\} - \mu_{2s}(m) - \beta_{2s}\mu_0(m) = 4s\Sigma_m{}'T_{2s-1}(\delta+1).$

One example will suffice to show the nature of these formulæ. Let $\zeta_r(n)$ denote the sum of the r^{th} powers of all the divisors of n. Then the value of

$$\Sigma(-1)^d\left[-1^5 + 3^5 - 5^5 + \ldots + (-1)^d(2d-1)^5\right],$$

in which Σ extends to all divisors d of n, is given by (2.11), and is $\tfrac{1}{4}\phi_5\{2\zeta(n),\ E\}$, viz.,

$$\tfrac{1}{2}\left[2^5\zeta_5(n) + 10.2^3E_2\zeta_3(n) + 5.2E_4\zeta_1(n)\right]$$
$$= 16\zeta_5(n) - 40\zeta_3(n) + 25\zeta_1(n).$$

12. We pass now to the manner in which sums of the kinds in §§ 6—9 enter into the arithmetical applications of elliptic functions and other quotients of the theta functions $\vartheta_\alpha(x)$, $\alpha = 0,\ 1,\ 2,\ 3.$ The trigonometric series for such

quotients as contain $\vartheta_1(x)$ or $\vartheta_2(x)$, or powers of these functions in the denominators will be sine or cosine developments according as the quotient is an odd or an even function of the argument, and in addition there will be a term involving $\tan x$, $\cot x$, $\sec x$, or $\operatorname{cosec} x$, or a power of these, which cannot be transformed so as to eliminate $1/\cos x$, $1/\sin x$ or their powers. For simplicity we may discuss only the case in which the term is $\tan cx$, c being a numerical constant. By suitably changing the variable this case is referred to that in which the term is $\tan x$. From any identity between this theta quotient containing the $\tan x$ term and other theta quotients, we derive by equating coefficients of like powers of q an identity between sums of sines or cosines of multiples of the argument, all products of sines or cosines being expressed as sums; and the latter identity will involve also a term in $\tan x$. If the coefficient of $\tan x$ does not vanish identically, it can be written as a sum of sines or cosines of multiples of the argument, so that $\tan x$ contributes terms of the form $\tan x \sin nx$ or $\tan x \cos nx$ to the identity. If n is odd these terms are at once reducible to the form (cf. the trigonometric identities in §§ 6–9) constant times ($\sec x$ plus sum cosines of multiples of the argument differing by 2), or a similar expression involving only sines. If n is arbitrary we replace n by $2n$, the procedure thereafter being similar to the foregoing.

Next, if $\tan x$ occurs to a power higher than the first, say the second, we write $\tan^2 x \sin mx = \tan x (\tan x \sin mx)$, express $\tan x \sin mx$ in the form first considered, multiply each term of the result by $\tan x$, and proceed as before with each of the products thus obtained. The case in which $\tan^n x$ occurs requires n repetitions of the process, and obviously the cases of $\cot^n x$, $\sec^n x$, $\operatorname{cosec}^n x$ may be treated similarly. In any of these cases the final trigonometric identity, after all reductions as indicated, which was obtained from the identity between theta quotients, involves sums of the form occurring in the right-hand members of the trigonometric identities in §§ 6–9. These final identities may also involve terms of the form $\tan x$ multiplied by a constant (independent of x); and it is easily seen that such terms must vanish identically.

If now in such a trigonometric identity the sines and cosines be expanded, and coefficients of like powers of x equated, the terms arising from that part of the identity which was contributed by $\tan^n x$, etc., will contain functions of the kind O, T, etc., of § 4. Obviously the sixteen forms that we have considered cover all possible cases. Each of them occurs frequently.

13. Instead of proceeding as outlined in the last of § 12, it is advantageous to use a much more powerful method which precipitates all of the arithmetical information implicit in the original theta identity into one general formula. Let the a, b denote integers. Then our final trigonometric identities are of one or other of the forms

$$a_0 + \Sigma_i a_i \cos n_i x = 0, \quad \Sigma_j b_j \sin n_j = 0;$$

and in these the terms arising from $\tan x$, etc., will have their arguments in arithmetical progression with common difference 2. Let $f(x)$ denote an even function of x which is single valued whenever x is an integer $\gtreqless 0$, and $g(x)$ an odd function of x, single valued whenever x is an integer $\gtrless 0$, and vanishing with x. Beyond these restrictions $f(x)$, $g(x)$ are arbitrary in the widest sense. Then it may be shown without difficulty that the above cosine identity implies

$$a_0 f(0) + \Sigma_i a_i f(n_i) = 0;$$

and the sine identity implies

$$\Sigma_j b_j g(n_j) = 0.*$$

From their definitions we may choose as special cases of f, g the following

$$f(x) = x^{2s}, \quad g(x) = x^{2s-1}.$$

These, substituted in the f, g formulæ above, are the forms which introduce the Bernoullian functions. Another choice of $f(x)$ leading to interesting results is $f(x) = x^{2s}$ when $x \equiv 0 \bmod n$, n a fixed integer, and $= 0$ in all other cases.

In the forthcoming paper, Part II., numerous f, g theorems for functions of one or more variables have been given as illustrations of the general processes considered in Part I., but no application is made to the special consequences involving Bernoullian functions. We shall therefore conclude with an example to show the nature of some of the simpler relations between divisors that may be found in this way.

14. A function $\psi_n(p, q)$, complementary to the ϕ defined in § 3, enables us to express many of these special consequences of elliptic theta identities in very simple and elegant form. It is given by

$$\psi_n(p, q) = (p+q)^n - (p-q)^n, \quad \psi_0(p, q) = 0;$$

* These are contained in a theorem concerning functions of n variables which is proved in Part I. (Sec. II.) of "Arithmetical Paraphrases", to appear shortly in the *Trans. Amer. Math. Soc.* (January, 1921).

so that its generators are

$$2\sin px \sin qx = -\cos\psi(p, q)\,x,$$
$$2\cos px \sin qx = \sin\psi(p, q)\,x.$$

15. As an illustration we shall take a formula relating to $\zeta_r(n)$, the sum of the r^{th} powers of all the divisors of n. In the paper on paraphrases it is shown that the trigonometric identity obtained by equating like powers of q in

$$\vartheta_2\vartheta_3\frac{\vartheta_0(x)}{\vartheta_1(x)} \times \vartheta_2\vartheta_3\frac{\vartheta_0(x)}{\vartheta_1(x)} = \left[\vartheta_2\vartheta_3\frac{\vartheta_0(x)}{\vartheta_1(x)}\right]^2$$

leads, upon proceeding as sketched in § 13, to

$$2\Sigma\left[f(d_1-d_2)-f(d_1+d_2)\right]$$
$$= \left[\zeta_1(m)-\zeta_0(m)\right]f(0) - 2\sum_{r=1}^{(\delta-1)/2} f(2r),$$

the Σ on the left extending to all odd positive integers d_1, d_2 which satisfy, for m fixed (and odd, by the notation of § 2)

$$m = 2^{a_1}d_1\delta_1 + d_2\delta_2,$$

and the Σ on the right referring to all divisors δ of m. If in this we put $f(x) = x^{2s}$, and write $\zeta_r'(n)$ for the sum of the r^{th} powers of the odd divisors of n, we have

$$\sum_{r=1}^{(m-1)/2} \psi_{2s}\{\zeta'(m-2r+1),\quad \zeta(2r-1)\} = \Sigma_m' T_{2s}(\delta+1);$$

and hence, on referring to (4 21),

$$2(2s+1)\sum_{r=1}^{(m-1)/2} \psi_{2s}\{\zeta'(m-2r+1),\ \zeta(2r-1)\}$$
$$= \phi_{2s+1}\{\zeta(m),\ R\} - \zeta_{2s+1}(m).$$

For $s = 1$ this gives

$$24\sum_{r=1}^{(m-1)/2} \zeta_1'(m-2r+1)\,\zeta_1(2r-1) = \zeta_3(m) - \zeta_1(m);$$

and as a numerical check, for $m = 7$, we have

$$24\left[\zeta_1'(6)\,\zeta_1(1) + \zeta_1'(4)\,\zeta_1(3) + \zeta_1'(2)\,\zeta_1(5)\right] = \zeta_3(7) - \zeta_1(7);$$

which is correct, since

$$\zeta_1'(6),\ \zeta_1'(4),\ \zeta_1'(2) = 4,\ 1,\ 1;$$
$$\zeta_1(1),\ \zeta_1(3),\ \zeta_1(5) = 1,\ 4,\ 6;\quad \zeta_3(7) = 344,\ \zeta_1(7) = 8.$$

ON THE ELLIPTIC FUNCTION TRANSFORMATION OF THE SEVENTH ORDER.

By *Arthur Berry.*

It is well known that Jacobi, after developing at the beginning of the *Fundamenta Nova* a purely algebraic theory of the transformation of elliptic functions, applied his method to carry out completely the transformations of orders 3 and 5, but did not deal in this way with any higher transformations. At a later point in the book (§ 20) he gave a general formula for the transformation of any odd order n, which may be called transcendental, inasmuch as it involved properties of elliptic functions, and in particular used elliptic functions of n^{th} parts of periods.

Cayley, in his "Memoir on the transformation of elliptic functions*", and in his book on elliptic functions, discussed algebraically the next higher transformation of odd order, that of order 7, but did not succeed in completing the solution. In a later paper, however†, after noticing that the equations must contain the solution of the problem of the cubic transformation as well as that of the transformation of order 7, he succeeded in overcoming the algebraic difficulties and gave the complete formulæ, using pure algebra only.

The algebraic work is, however, very laborious, and the object of this note is to show how it may be materially shortened and simplified.

The fundamental transformation formula being taken in the form

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{\alpha-\beta x+\gamma x^2-\delta x^3}{\alpha+\beta x+\gamma x^2+\delta x^3}\right)^2,$$

Jacobi's algebraical theory leads at once to Cayley's equations

$$u^{14}\alpha^2=v^2\delta^2,$$

$$u^6(2\alpha\beta+2\alpha\gamma+\beta^2)=v^2(2\beta\delta+\gamma^2+2\gamma\delta),$$

$$2\alpha\delta+2\beta\gamma+2\beta\delta+\gamma^2=u^2v^2(2\alpha\gamma+2\alpha\delta+\beta^2+2\beta\gamma),$$

$$2\gamma\delta+\delta^2=u^{10}v^2(\alpha^2+2\alpha\beta),$$

* *Phil. Trans.*, vol. clxiv. (1874) = *Coll. Math. Papers*, vol. ix. no. 578.
† *Phil. Trans.*, vol. clxix. (1878) = *Coll. Math. Papers*, vol. x. no. 692.

where u, v denote as usual $\sqrt[4]{k}$, $\sqrt[4]{\lambda}$, k and λ being the moduli.

The problem is to eliminate α, β, γ, δ between these four equations, so as to obtain an equation connecting u and v, the modular equation; and further to express $\alpha:\beta:\gamma:\delta$ as rational functions of u and v.

The essential idea which enabled Cayley to carry out the analysis was that, if $\alpha\delta-\beta\gamma=0$, the transformation formula reduces to

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{\alpha-\beta x}{\alpha+\beta x}\right)^2,$$

the formula for a cubic transformation, so that the solution of the equations must lead to the formulæ for the cubic transformation, as well as to those for the transformation of order 7.

The method which I use differs from Cayley's in rearranging the equations, before carrying out the elimination, so as to put *en évidence* the extraneous solutions belonging to the cubic transformation.

Ignoring an immaterial ambiguity of sign, we replace the first equation by

$$u^7\alpha=v\delta\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Using this equation we can write the other three equations

$$S_3\equiv\delta(2\alpha\beta+2\alpha\gamma+\beta^2)-\theta\alpha(2\beta\delta+\gamma^2+2\gamma\delta)=0\ldots\ldots\ldots(2),$$

$$S_2\equiv2\alpha\delta+2\beta\delta+2\beta\gamma+\gamma^2-\theta^2(2\alpha\gamma+2\alpha\delta+\beta^2+2\beta\gamma)=0\ldots(3),$$

$$S_1\equiv2\gamma+\delta-\theta^3(\alpha+2\beta)=0\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where θ is written for uv.

We have thus three homogeneous equations in α, β, γ, δ of orders 3, 2, 1; there are 6 sets of solutions, expressing the ratios of the variables as functions of θ, and we find that of these 6 sets 2 lead to the transformation of order 7, while the others (coinciding in pairs) give the cubic transformation.

From the known formulæ for the cubic transformation we have, in addition to

$$\Delta\equiv\alpha\delta-\beta\gamma=0\ldots\ldots\ldots\ldots\ldots(5),$$

the equation

$$\gamma-\theta\beta=0\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

We aim first at a linear combination of S_3, S_2, S_1, which is divisible by $\gamma-\theta\beta$. Writing

$$S_3\equiv2(1-\theta)(\beta+\gamma)\Delta+2(1-\theta)\beta\gamma(\beta+\gamma)+\beta^2\delta-\theta\alpha\gamma^2,$$

$$S_2\equiv2(1-\theta^2)\Delta+4(1-\theta^2)\beta\gamma+2\beta\delta+\gamma^2-\theta^2(2\alpha\gamma+\beta^2),$$

we easily find that such a combination is $S_3 - \beta S_2 + \beta^2 S_1$, and we accordingly replace (2) by

$$S_3 - \beta S_2 + \beta^2 S_1 \equiv (\gamma - \theta\beta)\,[2\,(1-\theta)\,\Delta$$
$$- \{\theta\alpha - (1-2\theta)\,\beta\}\,(\gamma - \theta\beta)] = 0 \ldots (7),$$

and somewhat similarly we can replace (3) by

$$S_2 - 2\beta S_1 \equiv 2\,(1-\theta^2)\,\Delta - \{2\theta^2\alpha + 4\theta^2\beta - \theta\beta - \gamma\}\,(\gamma - \theta\beta) \ldots (8).$$

Taking the solution $\gamma - \theta\beta = 0$ of (7), we have from (8) *either* $\theta^2 = 1$ (which is easily seen to lead to the degenerate case $k^2 = 1$, and can be ignored), *or* $\Delta = 0$, and these two equations lead to the cubic transformation. Rejecting this solution, (7) can be replaced by

$$2\,(1-\theta)\,\Delta - \{\theta\alpha - (1-2\theta)\,\beta\}\,(\gamma - \theta\beta) = 0 \ldots\ldots (9).$$

The equations (8) and (9) being homogeneous and linear in Δ and $\gamma - \theta\beta$ give *either* (5) and (6), leading again to the cubic transformation, *or*

$$2\theta^2\alpha + 4\theta^2\beta - \theta\beta - \gamma = (1+\theta)\,\{\theta\alpha - (1-2\theta)\,\beta\} = 0,$$

whence

$$\theta\,(1-\theta)\,\alpha - (1 - \theta + 2\theta^2)\,\beta + \gamma = 0 \ldots\ldots\ldots (10).$$

Thus, after rejecting the extraneous solutions, we have in place of the original equations (1) to (4) the three linear equations (1), (4), (10), and one of the quadratic equations (8) or (9).

Solving the three linear equations we find

$$\alpha : \beta : \gamma : \delta = 2\theta\,(1-\theta)\,(1-\theta+\theta^2) : \theta^4\,(2 - 2\theta + \theta^2) - u^8$$
$$: \theta^4 - (1 - 2\theta + 2\theta^2)\,u^8 : 2\,(1-\theta)\,(1-\theta+\theta^2)\,u^8 \ldots (11).$$

Substituting in (8) or (9), we have an equation in θ, u^8, which reduces readily to the usual modular equation

$$(1-u^8)\,(1-v^8) - (1-uv)^8 = 0 \ldots\ldots\ldots\ldots (12).$$

The equations (11) and (12) give the complete solution of the problem.

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THE DIHEDRAL ANGLES OF A TETRAHEDRON.

By *T. C. Lewis, M.A.*

1. There is a set of identities connected with the tetrahedron, most of which have been published by Professor Mathews*, which admit of easy proof.

2. If a, b, c be the edges of the base ABC, and d, e, f the edges from the vertex D to A, B, C respectively, write

$$a^2+d^2=A, \quad b^2+e^2=B, \quad c^2+f^2=C;$$

and let (BC) denote the angle between the faces meeting in the edge BC, and (ad) the angle between the edges a and d. Denote the areas of faces opposite A, B, C, D by α, β, γ, δ; and let V be the volume and R the circumradius of the tetrahedron. Then

$$\left.\begin{aligned} 2ad\cos(ad) &= B-C \\ 2be\cos(be) &= C-A \\ 2cf\cos(cf) &= A-B \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

3. Let the perpendiculars, h_a and h_d, from A and D upon BC meet that edge in P and Q. Then

$$2h_a h_d\cos(BC) = h_a^2 + h_d^2 - (d^2 - PQ^2),$$

therefore

$$8\alpha\delta\cos(BC) = 4(\alpha^2+\delta^2) - a^2d^2 + \tfrac{1}{4}(B-C)^2,$$

therefore

$$16\alpha\delta\cos(BC) = a^2(B+C-A) - b^2f^2 - e^2c^2 + b^2e^2 + c^2f^2 - a^2d^2:$$

similarly

$$\left.\begin{aligned} 16\beta\delta\cos(CA) &= b^2(C+A-B) - c^2d^2 - a^2f^2 + c^2f^2 + a^2d^2 - b^2e^2 \\ 16\gamma\delta\cos(AB) &= c^2(A+B-C) - a^2e^2 - b^2d^2 + a^2d^2 + b^2e^2 - c^2f^2 \\ 16\beta\gamma\cos(AD) &= d^2(B+C-A) - e^2f^2 - b^2c^2 + b^2e^2 + c^2f^2 - a^2d^2 \\ 16\gamma\alpha\cos(BD) &= e^2(C+A-B) - a^2c^2 - d^2f^2 + c^2f^2 + a^2d^2 - b^2e^2 \\ 16\alpha\beta\cos(CD) &= f^2(A+B-C) - a^2b^2 - d^2e^2 + a^2d^2 + b^2e^2 - c^2f^2 \end{aligned}\right\}$$

$$\ldots\ldots\ldots(2).$$

* See *Nature*, 29th August, 1918.

4. Write Q for $a^2b^2c^2+a^2e^2f^2+b^2d^2f^2+c^2d^2e^2$.

Let $I=16\alpha\delta\cos(BC)\times 16\beta\gamma\cos(AD)$

$$=a^2(B+C-A)-b^2f^2-e^2c^2+b^2e^2+c^2f^2-a^2d^2$$

multiplied by

$$d^2(B+C-A)-e^2f^2-b^2c^2+b^2e^2+c^2f^2-a^2d^2$$

$$=(B+C-A)\{a^2d^2(B+C-A)-Q+(b^2e^2+c^2f^2-a^2d^2)A\}$$
$$+(b^2f^2+e^2c^2)(e^2f^2+b^2c^2)$$
$$-BC(b^2e^2+c^2f^2-a^2d^2)+a^4d^4+b^4e^4+c^4f^4$$
$$-2a^2d^2(b^2e^2+c^2f^2)+2b^2e^2c^2f^2$$
$$=(B+C-A)\{a^2d^2(B+C-A)+b^2e^2(C+A-B)$$
$$+c^2f^2(A+B-C)-Q\}$$
$$+b^2e^2(B+C-A)(B-C)+c^2f^2(B+C-A)(C-B)-a^2d^2A(B+C-A)$$
$$+a^2d^2.BC+b^2e^2(C^2-BC)+c^2f^2(B^2-BC)$$
$$+a^4d^4+b^4e^4+c^4f^4-2b^2e^2c^2f^2-2c^2f^2a^2d^2-2a^2d^2b^2e^2$$
$$=(B+C-A).144V^2+a^4d^4+b^4e^4+c^4f^4-2b^2e^2c^2f^2-\ldots$$
$$+a^2d^2(A-B)(A-C)+b^2e^2(B-A)(B-C)+c^2f^2(C-A)(C-B).$$

Therefore

$$I=(B+C-A-4R^2).144V^2+a^2d^2(A-B)(A-C)$$
$$+b^2e^2(B-A)(B-C)+c^2f^2(C-A)(C-B).$$

So $\quad J=256\alpha\beta\gamma\delta\cos(CA)\cos(BD)$

$$=(C+A-B-4R^2).144V^2+a^2d^2(A-B)(A-C)+\ldots$$

and $\quad K=256\alpha\beta\gamma\delta\cos(AB)\cos(CD)$

$$=(A+B-C-4R^2).144V^2+\&c.$$

Thus

$$\left.\begin{aligned}I&=-288AV^2+H\\ J&=-288BV^2+H\\ K&=-288CV^2+H\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

5. The following identities are therefore at once established: From (1)

$$ad\cos(ad)+be\cos(be)+cf\cos(cf)=0\ldots\ldots\ldots(\text{i}).$$

From (2)

$$\alpha\delta \cos(BC)\, d^2 + \beta\delta \cos(CA)\, e^2 + \&c. = 27\, V^2 \ldots\ldots(\text{ii}).$$

From (3)

$$ad \cos(ad) \cos(BC) \cos(AD) + be \cos(be) \cos(CA) \cos(BD)$$
$$+ cf \cos(cf) \cos(AB) \cos(CD) = 0 \ldots\ldots\ldots(\text{iii}),$$

$$4\alpha\beta\gamma\delta \{\cos(AB) \cos(CD) - \cos(CA) \cos(BD)\}$$
$$= 9ad \cos(ad)\, V^2 \ldots\ldots(\text{iv}),$$

and two other similar identities.

6. If the tetrahedron is orthocentric $A = B = C$, and H in (3) reduces to

$$144\, V^2 (A + B + C - 4R^2),$$

where $A + B + C$ is the sum of the squares of the edges. Also

$$\cos(AB) \cos(CD) = \cos(BC) \cos(AD) = \cos(CA) \cos(BD).$$

END OF VOL. L.

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MESSENGER OF MATHEMATICS.

ON THE EARLY HISTORY OF THE SIGNS + AND − AND ON THE EARLY GERMAN ARITHMETICIANS.

By *J. W. L. Glaisher.*

Part I.

Introduction, §§ 1–5.

§ 1. It was at one time believed that the signs + and − were introduced into algebra by Stifel in his *Arithmetica Integra* of 1544*, but in 1864 De Morgan contributed to the Cambridge Philosophical Society a paper† in which he showed that they had been used by Widman in his *Rechenung* of 1489‡. This fact had however been previously noticed and pointed out by Drobisch§ in 1840; and his discovery of the signs had been noted by Gerhardt‖ in 1843 and by Cantor¶ in 1857.

De Morgan in his paper not only drew attention to the existence of the signs + and − in Widman's book, but he inferred from the mode of their occurrence that Widman or some predecessor had derived them 'from the warehouse', so that they had a commercial and not an arithmetical origin.

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* The first use of the signs + and − was attributed to Stifel by Hutton (*Phil. and Math. Dic.*, vol. i., article "Characters") and accepted by Klügel (*Wörterbuch*, vol. v., 1831, article "Zeichen"). Chasles (*Aperçu historique*, 2nd edn., 1875, p. 539), referring to Stifel's *Arithmetica Integra*, without explicitly stating that Stifel was the inventor of the signs, writes "On y trouve les signes +, − et le signe radical √" Libri says of the *Arithmetica Integra:* "This celebrated work is generally believed to be the first in which the signs +, − appear for *plus* and *minus*, of which he claims to be the inventor;" and he quotes from Stifel's *Deutsche Arithmetica* (1545) the passage in which he considers that this claim is made (Libri's sale catalogue of his library, nos. 593 and 594, p. 73. The sale took place April 25, 1861, and following days). This supposed claim is referred to in § 59.

† "On the early history of the signs + and −" (*Camb. Phil. Trans.*, vol. xi., 1871, pp. 203–212).

‡ "Behēde vnd hubsche Rechenung auff allen kauffmanschafft." (Leipzig, 1489).

§ "De Ioannis Widmanni Egerani . . . compendio arithmeticae mercatorum . . . scripsit Mauritius Guilielmus Drobisch" (Leipzig, 1840).

‖ Grunert's *Archiv.*, vol. iii., p. 291.

¶ *Zeitschrift für Math. und Phys.*, vol. ii., p. 366. In this paper Cantor states that in spite of Drobisch having found the signs in Widman's book, their invention was still generally attributed to Stifel.

§ 2. In connection with Stifel's writings upon which I had been engaged, I was led to examine Widman's *Rechenung* carefully, and I found myself unable to agree with most of De Morgan's conclusions and suggestions. The result of this examination and the inferences derived from it form the subject of the principal portion of Part I.

§ 3. It seemed likely that the arithmetics published subsequently to that of Widman might throw some light on the question whether his manner of using the signs + and − was due to the fact that they represented modes of expression which were employed in commercial transactions, or was derived from some other source, or originated in himself; and with this object I examined the principal arithmetics and algebras published in Germany up to 1550 and noted the occurrences of the signs + and −, or the words plus and minus, or mehr and weniger, both in the explanations and in the examples, paying special attention to the class of questions in which Widman used + and −. The results of this examination are contained in Part II.

§ 4. In the course of the work connected with Parts I. and II. I met with references to the writings of Wappler, Curtze and others, and when these two Parts were practically completed, I turned my attention to their investigations, which mainly related to manuscripts earlier than Widman's book, some of which had actually been in Widman's possession. The manuscripts give valuable information with respect to the previous history of the signs + and − and to the use of the words plus and minus. The results derived from the study of these papers are contained in Part III. I have thought it desirable to keep this Part quite distinct and to print Parts I. and II. as they were originally written*, for they were derived entirely from an examination of Widman's *Rechenung* and its successors before I had read the manuscripts referred to in Part III. The conclusions in Parts I. and II. are in striking agreement with those which resulted from an examination of the earlier writings.

§ 5. The preceding paragraphs give a general idea of the contents of the three Parts, as distinguished from each other, but in each Part other questions having some relation to the

* When writing Parts I. and II. I had not seen Tropfke's "Geschichte der elementar-mathematik" (Leipzig, 1902), and the references to this work have been added subsequently.

main questions are included, as for example, Widman's use of the cossic signs in Part I., the words fusti and tara in Part II., and matters connected with the early history of algebra in Part III.

It will be seen that although the history of the signs + and − forms the main subject of the paper there are a number of other questions relating to the arithmetic and algebra of the fifteenth and the first half of the sixteenth century which are also referred to.

With respect to the origin of the signs + and −, I consider that all the evidence shows that they were derived from algebra and not from commerce. This opinion resulted from a study of Widman's book itself, and it is abundantly confirmed by the writings quoted in Part III.

Printed arithmetics of the 15*th century*, §§ 6–7.

§ 6. Before describing the manner in which the signs + and − made their appearance in Widman's *Rechenung*, it is convenient to mention the principal printed arithmetics of the 15th century: these are

1° The Treviso Arithmetic (Treviso, 1478).
2° The Bamberg Arithmetic (Bamberg, 1483).
3° Borgi (Venice, 1484).
4° Widman (Leipzig, 1489).
5° Calandri (Florence, 1491).
6° Pellos (Turin, 1492).
7° Paciolo (Venice, 1494).*

An account of the anonymous Treviso Arithmetic, 1478, is given by Eugene Smith in *Rara Arithmetica*†, pp. 3–7, where there is also a facsimile reproduction of its first page and of other pages.

The Bamberg Arithmetic (1483), believed to have been written by Ulrich Wagner, is described by Cantor in vol. ii.

* I write Paciolo in preference to Pacioli or Paciuolo. For a long time Pacioli was much the most usual form, though Paciolo also occurred. In 1889 Staigmuller introduced the spelling Paciuolo which was adopted by Cantor, Eneström, Eugene Smith, and some German writers. My reasons for preferring the spelling Paciolo will be given in a separate paper, as they would occupy more space than can be afforded in a note.

† "Rara Arithmetica. A catalogue of the arithmetics written before the year MDCI with a description of those in the library of George Arthur Plimpton of New York by David Eugene Smith of Teachers College Columbia University" (Boston and London, 1908). I have found this work most useful and valuable. A number of the title pages of the arithmetics described in it are reproduced in facsimile, and whenever I shall have occasion to mention a book of which the title page is reproduced in *Rara Arithmetica* I shall merely give a short title and a reference to the page where the reproduction appears.

of his *Vorlesungen*, 2nd edition, pp. 221–227*, and by Unger, *Die Methodik*, pp. 37–40†. Both Cantor and Unger also describe the fragments of an earlier arithmetic, probably by the same author, printed at Bamberg in the previous year.

Borgi's Arithmetic (1484) is a good mercantile treatise. It bears the title "Qui comenza la nobel opera de arithmethica ne la qual se tracta tute cosse amercantia pertinente facta e compilata per Piero borgi da veniesia". A reproduction of the first page is given in *Rara Arithmetica*, p. 18. There was a second edition in 1488. The title of Widman's book (1489) is given in § 8.

Calandri's Arithmetic (1491) gives the fundamental processes of calculation, and the rule of three with applications to commercial questions. There are a great many fancy or puzzle problems. None of the commercial questions are of much complexity. The title page and other pages are reproduced in *Rara Arithmetica* (pp. 46–49). The first page of the book commences "Philippi Calandri ad nobilem et studiosuꝫ Julianum Laurentii Medicē de arimethrica opusculū".

Pellos's Arithmetic (1492) is much more complete and practical and it contains numerous mercantile applications. The title is "Sen segue de la art de arithmeticha et semblātment de ieumetria dich ho nominatꝫ Cōpendiō de lo abaco". The title page and one other page are reproduced in *Rara Arithmetica* (pp. 50–52)‡.

Paciolo's work (1494) is of a very different character. It is a large folio volume and probably contains a full account of all that was then known of arithmetic and algebra. There are numerous commercial questions, some of which are very complicated: also explanations of commercial terms and methods. It is a fine and comprehensive work. The title page is reproduced on p. 55 of *Rara Arithmetica*.

§ 7. It thus appears that before Widman's book only three practical arithmetics had appeared in print, two in Italy (in Italian) and one in Germany (in German). No printed algebra had appeared, Paciolo's *Summa* (1494) being the first book which contained algebra.

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* "Vorlesungen über geschichte der mathematik von Moritz Cantor", vol. ii. (Leipzig, 1900). All my references will be to the second edition, which has since been reprinted without alteration.

† "Die methodik der praktischen arithmetik in historischer entwickelung . . . von Friedrich Unger" (Leipzig, 1888).

‡ I mention only practical arithmetics in which the processes of addition, subtraction, multiplication, division, and the rule of three are explained. Such books are necessarily to some extent commercial. I omit the few arithmetics of this period on the Boethian system: these are described by Eugene Smith in *Rara Arithmetica*.

Borgi's Arithmetic was published five years before Widman's, and Paciolo's *Summa* five years after. These works afford abundant evidence of the state of arithmetic and its applications in Italy at the period when Widman was engaged upon his *Rechenung*.

Widman's Rechenung of 1489, §§ 8–10.

§ 8. Widman's book was published at Leipzig in 1489. Its title is "Behēde vnd hubsche Rechenung auff allen kauffmanschafft": and the colophon is "Gedruckt In der Furstlichen Stath Leipczick durch Conradū Kacheloffen Im 1489 Iare".

The next two editions (to which I shall occasionally refer for comparison with the original edition) were published at Pforzheim in 1500 and 1508. Later editions were published at Hagenau in 1519 and at Augsburg in 1526*.

The book consists of the rules of arithmetic, fractions, proportions, progressions, rule of three, and applications to many commercial questions. There are also fancy and puzzle questions and the rule of false. At the end there is a geometry.

In the original edition of 1489 the leaves are not numbered, and only the first page of each sheet has a signature. I shall therefore follow Boncompagni in referring to this edition by leaf and page. The book contains 236 leaves: in numbering them, that containing the title page is taken to be the first. The connection between the pages and signatures is as follows: 9, b; 17, c; 25, d; 34, e; 42, f; 50, g; 58, h; 66, i; 74, k; 82, l; 90, m; 98, n; 106, o; 114, p; 122, q; 130, r; 138, s; 146, t; 154, v; 162, x; 170, y; 178, z; 186, A; 194, B; 202, C; 210, D; 218, E; 226, F; 234, G. In referring to pages I denote the recto of a leaf simply by its leaf number and the verso by this number accented; thus p. 87 denotes the recto of f. 87 and p. 87′ denotes the verso of f. 87.† In the editions of 1500 and 1508 the leaves are numbered.

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* Widman's *Rechenung* and its various editions are the subject of a comprehensive paper by Boncompagni, entitled "Intorno ad un trattato d'aritmetica di Giovanni Widmann di Eger" in vol. ix of his *Bullettino di Bibliografia e di storia* (pp. 188–210); Boncompagni does not mention the edition of 1500. An account of the contents of Widman's book is given by Unger, pp. 40–42, and by Cantor, vol. ii., pp. 228–236.

† This method of distinguishing between recto and verso is followed also throughout this paper in referring to other books, whether the leaf is denoted by its number or by its signature and number, *e.g.* E vi denotes the recto of the leaf E vi, and E vi′ its verso.

§ 9. The sign + first occurs in the question*: "Als in diesē exēpel 16 ellñ pro 9 fl $\frac{1}{3}$ vñ $\frac{1}{4}+\frac{1}{5}$ eynss fl wy kūmē 36 ellñ machss alsso Addir $\frac{1}{3}$ vñ $\frac{1}{4}$ vñ $\frac{1}{5}$ zu sāmen kumpt $\frac{47}{60}$ eynss fl Nu secz vñ machss nach der regl vñ kūmē 22 fl $\frac{1}{80}$ eynsz fl dz ist gerad 3 hlr in gold†".

Here the question is: "If 16 ells are bought for 9 florins and $\frac{1}{3}$ and $\frac{1}{4}$ and $\frac{1}{5}$ of a florin what will 36 ells cost?" He adds $\frac{1}{3}$ and $\frac{1}{4}$ and $\frac{1}{5}$ obtaining $\frac{47}{60}$: thus 16 ells cost $9\frac{47}{60}$ fl and therefore the cost of 36 ells is given by finding the fourth proportional to 16, $\frac{587}{60}$, 36, which is $22\frac{1}{80}$ fl or 22 fl 3 heller, as stated by Widman. In this question + is clearly a misprint, or is due to a slip in writing, and its occurrence is unintentional. This is evident from the use of vñ in its place immediately afterwards‡. It will be seen that the question is an exercise in the treatment of fractions and the use of the rule of three.

§ 10. Widman's next question§ is: "Itm̄ 9 ellñ pro 6 fl $\frac{1}{3}$ eynss fl vnd $\frac{1}{2}$ eynss drittñ vō $\frac{3}{4}$ von eynē halbñ dritten eynsz fl wie kummen 11 ellñ $\frac{1}{8}$ machss alszo wart was $\frac{1}{2}$ dritten sey das ist $\frac{1}{6}$ Darnach wart wass $\frac{3}{4}$ von $\frac{1}{6}$ sey das ist $\frac{1}{8}$ Nu addir $\frac{1}{3}$ vnd $\frac{1}{6}$ vnd $\frac{1}{8}$ zusamen vnd kumpt $\frac{5}{8}$ vnd secz also $\frac{9}{1} \times \frac{53}{8} - \frac{89}{8}$ Machss nach der Regel vnd küpt 8fl 3ss 9helr $\frac{5}{12}$".

The solution of this question shows that 'vō' which precedes $\frac{3}{4}$ is a misprint for 'vñ'. In the question 9 ells cost 6 florins and $\frac{1}{3}$ of a florin and $\frac{1}{2}$ of $\frac{1}{3}$ of a florin and $\frac{3}{4}$ of $\frac{1}{2}$ of $\frac{1}{3}$ of a florin: that is, they cost 6 florins and $\frac{1}{3}$ and $\frac{1}{6}$ and $\frac{1}{8}$ of a florin, which is $6\frac{5}{8}$ florin. Thus the first three terms of the proportion are 9, $\frac{53}{8}$, $\frac{89}{8}$, so that the result is $\frac{4717}{576}$ florins which, as Widman states, is 8 fl 3 ss $9\frac{5}{12}$ hlr.

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* Widman, p. 87. For the word *pro* Widman here and elsewhere uses the customary abbreviation of a p with the lower part of the loop extended across the stem. Widman's abbreviation for florins, which I write as fl, was, I think, merely intended to be an f, followed by a loop or flourish. Similarly I shall write ct for centners and gr for groschen, although I think the marks used by Widman and others were merely meant for c and g, followed by a loop or flourish, and that it would here have been more consistent if I had represented the abbreviations by fe, ce, ge, or fρ, gρ, gρ. Widman follows the general custom of the time in writing the fractions after the denomination, *e.g.* 9 fl $\frac{1}{3}$ instead of $9\frac{1}{3}$ fl. In general, I shall place the fraction before the denomination, except when I am quoting the actual words.

† In 'gold reckoning' a florin is 20 shillings and a shilling is 12 heller. These denominations are denoted by fl, ss, hlr.

‡ De Morgan's comment, in quoting this first occurrence of +, is: "Here the + is probably a sign to which the writer was accustomed, but which he did not then intend to print, since *und* occurs both before and after" (*Camb. Phil. Trans.*, vol. xi., p. 205).

§ Widman, p. 87.

Here as before the question is an exercise in the treatment of fractions and the rule of three, and Widman uses 'vnd' for addition.

Widman's question (on figs) in which + and – are first used, § 11.

§ 11. His next question, which relates to the purchase of barrels of figs, is the one in which + and – are used and their meaning defined. This question is as follows:*

"Veygen.

"Itm̄ Eyner kaufft 13 lagel veygen vn̄ nympt ye 1 ct pro 4 fl $\frac{1}{2}$ ort Vnd wigt itliche lagel als dan hye nochuolget. vn̄ ich wolt wissen was an der sum brecht

	4 + 5	Wiltu dass
	4 – 17	wyssen ader†
	3 + 36	dess gleichn̄
	4 – 19	Szo sum-
	3 + 44	mir die ct
	3 + 22	vnd lb vn̄
Czentner	3 – 11 lb	was – ist
	3 + 50	dz ist mi⁹
	4 – 16	dz secz besū
	3 + 44	der vn̄ wer
	3 + 29	dē 4539
	3 – 12	lb (So du
	3 + 9	die ct zcu lb

gemacht hast vnnd das + das ist mer dar zu addirest) vnd 75 min⁹ Nu solt du fur holcz abschlahn̄ albeg fur eyn lagel 24 lb vn̄ dz ist 13 mol 24· vn̄ macht 312 lb dar zu addir dz – dz ist 75 lb vnnd werden 387 Die subtrahir vonn 4539 Vnnd pleybn̄ 4152 lb Nu sprich 100 lb das ist 1 ct pro 4 fl $\frac{1}{8}$ wie kummen 4152 lb vnd kūmen 171 fl 5 ss 4 hlr $\frac{4}{5}$ Vn̄ ist recht gemacht.‡

* Widman, p. 87′.

† In the original an abbreviation is printed for 'der', which I expand, writing ader at length. As already mentioned, I write ct for Widman's symbol for centners.

‡ This question has also been reprinted by De Morgan (*Camb. Phil. Trans.*, vol. xi., p. 205) and by Boncompagni (*Bullettino*, ix., p. 205). In De Morgan's reprint he puts 'pr 4 fl $\frac{1}{2}$ (misprint for $\frac{1}{8}$) ort', but there is no misprint, for an ort is a quarter, so that 4 fl $\frac{1}{2}$ ort is 4 $\frac{1}{8}$ fl.

A nearly literal translation is as follows. "A person buys 13 barrels of figs and receives 1 centner for $4\frac{1}{8}$ florins, and the weight of each barrel is as follows: 4 ct + 5 lb, 4 ct − 17 lb, 3 ct + 36 lb, 4 ct − 19 lb, 3 ct + 44 lb, 3 ct + 22 lb, 3 ct − 11 lb, 3 ct + 50 lb, 4 ct − 16 lb, 3 ct + 44 lb, 3 ct + 29 lb, 3 ct − 12 lb, 3 ct + 9 lb; and I would know what they cost. To know this and the like, sum the ct and lb and what is −, that is minus, set aside, and they become 4539 lb (if you bring the centners to lb and thereto add the +, that is more) and 75 minus. Now you must subtract for the wood 24 lb for each barrel and 13 times 24 is 312 to which you add the −, that is 75 lb and it becomes 387 which subtract from 4539 and there remains 4152 lb. Now say 100 lb that is 1 ct for $4\frac{1}{8}$ fl, what do 4152 lb come to, and they come to 171 fl 5 ss $4\frac{4}{5}$ hlr which is right".

The sentence in which the signs are explained may be paraphrased as follows. Add the ct and lb, setting aside the lb which have −, which is minus, and the result is 4539 lb (when you have converted the ct into lb and added the lb which have +, which is more), and there are 75 lb minus.

De Morgan's 'warehouse' theory of the origin of the signs, § 12.

§ 12. De Morgan was led by this question to suggest that the signs + and − came 'to the arithmetician from the warehouse', and this opinion was confirmed by two other questions (relating to pepper and soap) which he quotes and which will be referred to in §§ 16 and 18. De Morgan's words are "The chests are weighed by centners of 100 lb. each, and the run of the chests being from 3 to 4 centners, the obvious plan is to put three or four centners into the scale of the weights, and make the balance by pounds in that scale, or in the scale of the goods, as wanted. The first chest wants *more*, and is 4 *c.* 5 *lb*: the second wants *less*, and is 4 *c.* all but 17 *lb.* It will fully appear that + and − are *plus* and *minus*, from which addition and subtraction follow by *inference.* This presentation of data is not the doing of the arithmetician, as such: it seems to be served up direct from the warehouse. It may be suspected that the nearest number of centners would be placed in the scale by guess, and separate record would

* In a paper attached to De Morgan's copy of Widman's *Rechenung*, now in the library of the University of London, he has written in reference to this question: "On 85 is the problem from the statement of which I derived the suspicion that + and − were originally marks used in the warehouse to denote overplus or under minus of weight. A. De Morgan, Nov^r 20, 1864". (Boncompagni's *Bullettino*, vol. ix., p. 191).

be made of the overplus or underminus—if this last word may be allowed. It may be suspected that + and − were warehouse marks, annexed to the entry of the weights for distinction, perhaps painted or chalked on the chests".*

Views of Drobisch, Gerhardt, &c., §§ 13–15.

§ 13. Drobisch†, who was the first to point out the occurrence of the signs + and − in Widman, says that he uses them in passing as if they were sufficiently known, merely remarking "was − ist das ist minus vnd das + das ist mer".

Gerhardt considered that the signs + and − were derived from mercantile practice, where they were in general use, and he says that in Widman their application is isolated and not general‡.

Treutlein did not express an opinion on the origin of the signs, merely accepting Drobisch's inference that they were already known in Widman's time§.

§ 14. Thus both De Morgan and Gerhardt were of opinion that Widman's *Rechenung* indicated a commercial origin of the signs, De Morgan making the definite suggestion that they arose in the warehouse and passed directly to arithmetic as data, while Gerhardt considered that Widman's use of them was restricted.

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* *Camb. Phil. Trans.*, vol. xi., p. 206.

† "Multo magis vero invat annotare, *apud Widmannum nostrum primum non solum inter Germanos, sed inter omnes gentes signorum additionis et subtractionis + et − plus et minus usum observari.* Sed utitur iis praetereundo, quasi de re iam satis nota loqueretur, dicens: '*was − ist das ist minus vnd das + das ist mer* (*De . . . Widmanni . . . compendio*, p. 20).

‡ Gerhardt, besides the paper referred to in § 1 (which relates to the history of algebra in Italy), contributed two papers to the Berlin Academy in 1867 and 1870, in which the signs + and − are referred to, and he subsequently published his "Geschichte der Mathematik in Deutschland" (Munich, 1877). The following are the passages which relate to the origin of the signs:

"Die Zeichen + und − (deren Ursprung nach meinem Dafürhalten aus der kaufmännischen Praxis herzuleiten ist) finden sich zuerst in dem Rechenbuch des Joh. Widmann von Eger, das im Jahre 1489 zu Leipzig erschien, aber sie kommen darin nur vereinzelt, nicht überall zur Anwendung" (*Monatsberichte der Berl. Akad. für* 1867, p. 53).

"Als ein besonderer Vorzug von Widman's Rechenbuch muss noch hervorgehoben werden, dass in demselben zum ersten Mal die Zeichen + und − vorkommen; die Art und Weise der Einführung scheint darauf hinzudeuten, dass diese Zeichen im kaufmännischen Verkehr üblich waren" (*Gesch. der Math.*, p. 36).

§ *Zeitschr. für Math. u. Phys.*, vol. xxiv., Supp. p. 29. Drobisch had mentioned that Peurbach, in explaining the rule of false, had spoken of signs of addition and subtraction, but he considered that they were not + and − but merely signs which were left to the reader. Treutlein took the contrary view, and thought that they did refer to + and −, and he found a confirmation of this belief in their occurrence in a manuscript described by Gerhardt in the *Monatsber. der Berl. Akad.* for 1870 (p. 143), which was then believed to be of the 15th century, but which has since been shown to belong to the 16th century, and therefore to be subsequent to Widman's book (Cantor, *Gesch. der Math.*, vol. ii., 2nd ed. p. 240).

In order to decide to what extent these views are justified it is necessary to examine not only the way in which the signs first arise (in the question about the figs), but also their use and mode of treatment in all the other questions in which they occur.

§ 15. In connection with the question about the figs (§ 11) the following points seem to me to be worth attention. (1) The wording of the question distinctly suggests that the weights arise in the forms 4 ct + 5 lb, &c., as the result of the weighings, but of course there is nothing to show that + and − were the actual marks used for plus and minus in the weighing-room. (2) Although the signs are used in the list of weights before they are defined, their meaning is explained immediately afterwards at the beginning of the solution. (3) The subtraction of 24 lb for the weight of the wood in each barrel, though really part of the question, is only mentioned in the course of the solution, so that there is little significance in + and − being used before they are explained. (4) Widman treats the weights algebraically, *i.e.*, he adds all the plus terms and all the minus terms instead of finding by addition or subtraction the weight of each barrel, and then adding these weights. It is also noticeable that he treats the weight of the wood of the barrels as a minus quantity, adding it to the minus sum which had already been obtained.

Other questions of Widman's in which + and − occur in the data, §§ 16–20.

§ 16. The example which immediately follows the fig question relates to pepper, and is as follows.

"Pheffer:

"Itm̄ 1 sack pfeffer wigt 2 ct $\frac{1}{2}$ − lb* vn̄ ist 1 lb kaufft wordē pro 8 ss − 3 hlr vn̄ sol fur den sack abschlahn̄ 3 lb + $\frac{3}{4}$ was kost das alles Machss alsso subtrahir die 9 lb vnd 3 lb $\frac{3}{4}$ von 250 pleyben 237 lb $\frac{1}{4}$ Darnach subtrahir auch 3 hlr von 8 ss pleybn̄ 93 hlr Nu secz also 1 lb pro 93 hlr wie 237 lb $\frac{1}{4}$ machss nach der Regel szo kummen 91 fl 18 ss 8 hlr $\frac{1}{4}$ &c.".

Here the sack of pepper weighs $2\frac{1}{2}$ ct − 9 lb and 1 lb of pepper costs 8 ss − 3 hlr and 3 lb + $\frac{3}{4}$ is to be subtracted for the sack. To find the cost of the pepper Widman subtracts

* Widman, p. 88′. The weight '2 ct $\frac{1}{2}$ − lb' should be '2 ct $\frac{1}{2}$ − 9 lb'. It is correctly printed in the editions of 1500 and 1508. The heading is given as 'Pfeffer' (not 'Pheffer') in these editions.

9 lbs and $3\frac{3}{4}$ lbs from 250 lbs, leaving $237\frac{1}{4}$ lbs; he subtracts 3 hlr from 8 ss, leaving 93 hlr, and as 1 lb of pepper costs 93 hlr, therefore $237\frac{1}{4}$ lbs cost 91 fl 18 ss $8\frac{1}{4}$ hlr.

In this question the sign − is used in expressing not only the weight but the money. The sign − in connection with money could not arise in the warehouse, for even if the 3 hlr were a rebate of some kind on the 8 ss it would be a matter that concerned the accounts in the office and not the procedure in the warehouse.

The introduction of + in the weight 3 lb + $\frac{3}{4}$ was unnecessary, but Widman may have wished to display its use in connection with −.

§ 17. I now give some account of all the other questions in which + and − are used. There are a good many questions in which they do not occur: these I pass over, confining myself to those in which Widman uses either + or − or both.

In the next two questions of this class, which relate to saffron and grapes, signs are used only in connection with money.

In the first of these questions* a man buys 10 ct 11 lb of saffron for 2360 fl, and sells it for 3 fl $-\frac{1}{2}$ ort† the lb, after paying $94\frac{1}{2}$ fl for expenses. How much does he gain or lose? This is solved as follows. Adding $94\frac{1}{2}$ fl to 2360 fl we have $2454\frac{1}{2}$ fl as the cost of 10 ct 11 lbs, so that 1 lb has cost 2 fl 8 ss $6\frac{678}{1011}$ hlr‡. Subtracting this from 3 fl $-\frac{1}{8}$, the difference is 8 ss $11\frac{333}{1011}$ hlr, which is what he gains on each lb. Here in the working 3 fl $-\frac{1}{8}$ is in effect replaced by 2 fl 17 ss 6 hlr.

In the second question§ a man buys 4 barrels of grapes weighing 9 ct 12 lbs, and 1 ct costs 6 fl − 1 ort $\frac{1}{2}$. How much does he pay? In this case Widman merely gives the result 51 fl 6 ss. As an ort is $\frac{1}{4}$, 1 ort $\frac{1}{2}$ is $\frac{1}{4}$ and $\frac{1}{2}$ of $\frac{1}{4}$, that is $\frac{3}{8}$, and 6 fl $-\frac{3}{8}$ fl is 5 fl 12 ss 6 hlr, which is the cost of 100 lbs.

§ 18. The next two questions, relating to oil and soap, are similar to the question about figs (§ 11) as regards the

* Widman, p. 92.

† In the edition of 1508 this is misprinted 3 fl $\frac{1}{2}$ ort.

‡ There is an error in the working, Widman giving this quantity as 2 fl 8 ss 6 hlr $\frac{298}{1011}$. He then proceeds: "Nu subtrahir von 3 fl $-\frac{1}{2}$ orth macht das er an eynē lb gwint 8 ss 11 hlr $\frac{719}{1011}$ eynsz hlr Nu sprich 1 lb giebt mir gwinss 8 ss 11 heller $\frac{713}{1011}$ Was gebē mir 10 ct 11 lb machss nach der Regel so kumpt das facit". In the editions of 1500 and 1508 the values of the fractions are given correctly, but in all the editions 1011 is sometimes printed 1101.

§ Widman, p. 93. In the editions of 1500 and 1508 the cost is given as '6 fl − anderhalb ort': anderhalb is $1\frac{1}{2}$.

expression of the data, *i.e.* the gross weights are given in centners and pounds, connected by the signs + and −.

In the first of these questions* a man buys three barrels of oil, of which the weights are given as '2 ct 18 lbs', '3 ct − 32 lbs', '3 ct + 5 lbs'. The wood amounts to 9 lb in each centner, and the cost of the oil is 1 gr 9ϑ†. How much does he pay? Widman does not give the solution, but says the question is to be solved in the same manner as the question about figs, and he merely states the result‡.

The sum of the centners and pounds with the + sign is 823 lbs, from which 32 lb is to be subtracted, leaving 791 lb as the weight of the three barrels of oil. From this 9 per cent., viz., $71\frac{19}{100}$ lb is to be subtracted, leaving $719\frac{81}{100}$ lb as the weight of the oil, which at 21ϑ the lb gives 59 fl 20 gr $8\frac{1}{100}$ϑ.

In this example the method used in the fig question cannot be followed exactly, for the gross weight of the barrels of oil must be obtained before the weight of the wood can be calculated.

The next question§ is of the same kind. A man buys four barrels of soap weighing 4ct − 63lb, 3ct + 24lb, 3ct − 2lb, 4 ct + 1 lb. The wood amounts to 12 lb in each centner, and the soap costs 5 fl 18ss 1 hlr the lb. Widman gives the result as 70 fl 13 ss $2\frac{82}{125}$ hlr, which is correct.

§ 19. In the remaining two questions of this class the minus sign occurs in connection with money only.

The first‖ of these questions relates to cloves. A man buys 2781lb¶ of cloves and stalks (fusti), of which 13lb out of every 100lb are stalks. The cloves cost 11 ss 3 hlr the lb, and the stalks 2 ss − 3 hlr. How much does he pay? Widman's solution is: As 100 lb produces 13 lb of stalks, therefore 2781 lb produce $361\frac{53}{100}$ lb, and therefore there are $2419\frac{47}{100}$ lb

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* Widman, p. 93.

† I use ϑ for the variant of *d*, which denotes 'pfennige'. The upper portion is correctly represented by ϑ, but the lower portion should be bent round to form a loop.

‡ "Machss gleicher weiss alsz obn̄ mit den veygē vn̄ kumpt 59 fl 6 gr 9ϑ 1 hlr $\frac{4}{5}$". This result is not correct: a florin is 21 groschen, a groschen 12 pfennige, and a pfennig is two heller: the true result is that given in the text. I find that Widman's result would be obtained if the wood in each centner were taken to be 10 lb instead of 9 lb. In the edition of 1500 the money is expressed in florins, shillings and heller, the price of the oil being 1 ss 9 hlr, and the result being given as 62 fl 19 ss $8\frac{1}{100}$ hlr, which is correct. In the edition of 1508 the weights of the barrels are given as 2 ct 18 lb, 3 ct − 32 lb, and 4 ct + 5 lb, and the result is given as 70 fl 18 ss $11\frac{1}{100}$ hlr, which is correct.

§ Widman, p. 93′. ‖ *Id.*, p. 96.

¶ Misprinted 278. It is correctly printed in the editions of 1500 and 1508.

of cloves, which at 11 ss 3 hlr the lb come to 1360 fl 19 ss $\frac{9}{20}$ hlr, and the $361\frac{53}{100}$ lb of stalks at 21 hlr the lb come to 31 fl 12 ss $8\frac{13}{100}$ hlr. Thus the total cost is 1392 fl 11 ss $8\frac{29}{50}$ hlr.

The other question* relates to various kinds of corals, weighing $69\frac{1}{3}$ lb, $59\frac{1}{4}$ lb, $49\frac{1}{2}$ lb, $39\frac{1}{4}$ lb, $29\frac{1}{8}$ lb, 19 lb, the respective prices per lb of which are, in florins, $6+\frac{1}{8}$, $5+\frac{1}{8}$, $4+\frac{1}{8}$, $3+\frac{1}{8}$, $3-\frac{3}{4}$, $2+\frac{1}{8}$†. The respective values are given by Widman as 424 fl 13 ss 4 hlr, &c., the total amount being 1161 fl 1 ss $5\frac{1}{2}$ hlr.

§ 20. In only one of these eight questions (viz., that relating to pepper) does the minus sign connect centners and pounds as well as shillings and heller: in three (relating to saffron, grapes, cloves and corals) it connects florins and fractions of a florin, or shillings and heller. Among the weights, those with the minus sign are 4 ct − 17 lb, 4 ct − 19 lb, 3 ct − 11 lb, 4 ct − 16 lb, 3 ct − 12 lb (figs), $2\frac{1}{2}$ ct − 9 lb (pepper), 3 ct − 32 lb (oil), 4 ct − 63 lb, 3 ct − 2 lb (soap). It will be noticed that in one of these weights $2\frac{1}{2}$ ct occurs, and that in 4 ct − 63 lb, the lbs exceed half a centner.

In money the amounts which have the minus sign are 8 ss − 3 hlr (pepper), 3 fl − $\frac{1}{2}$ ort (saffron), 6 fl − 1 ort $\frac{1}{2}$ (grapes), 2 ss − 3 hlr (cloves), 3 fl − $\frac{3}{4}$ fl (corals). Thus the deductions are 3 hlr, $\frac{1}{8}$ fl, $\frac{3}{8}$ fl, 3 hlr, and $\frac{3}{4}$ fl. It is possible that these deductions may have arisen from some kind of rebate, but this seems unlikely, for in the question about cloves, the cloves cost 11 ss 3 hlr, and the stalks 2 ss − 3 hlr, and in the question about corals, in five cases a fraction is added and in only one is it subtracted, and then it is as large as $\frac{3}{4}$. It seems to me much more probable that the + and − terms were introduced merely as exercises in the use of these signs.

Other questions of Widman's in which + *and* − *occur*, § 21.

§ 21. I now proceed to consider the other questions in which Widman uses the signs + and −.

(1) The expression $\frac{3}{5}+\frac{2}{5}$ occurs in the solution of an example‡ in which wine at 5 fl is mixed with wine at 10 fl, and the question is to determine the right proportion in order that the mixture may be worth 7 fl. It is found that the composition must be $\frac{3}{5}+\frac{2}{5}$, *i.e.* that there is to be $\frac{3}{5}$ of the cheaper wine and $\frac{2}{5}$ of the dearer.

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* Widman, p. 155.

† This is misprinted $2+\frac{1}{4}$. The fraction should be $\frac{1}{8}$, as appears from the result given, viz. 40 fl 7 ss 6 hlr. The correct fraction $\frac{1}{8}$ is given in the editions of 1500 and 1508.

‡ Widman, p. 109.

(2) In a similar question*, where there are four wines at 20₰, 15₰, 10₰, 8₰, and a mixture worth 12₰ is to be made. The result is that 6 parts are to be taken 'vonn dem pro 20₰ + 15₰' (that is, of each of the two more expensive wines) and 11 parts 'der geringern zweyer weyn'.

(3) A man buys 6 eggs − 2₰ for 4₰ + 1 egg. How much does an egg cost?† The result is found to be $1\frac{1}{5}$₰.

(4) The compound interest of 20 florins for 2 years is 30 florins. What is it for 1 year? The answer is given as the square root of 600 − 20 (die wurzel von 600 − 20)‡.

(5) In another question§ on interest 1 florin has produced 1 fl + 36₰, and the question is to find what would produce 50 fl. Here + merely adds pence to florins.

(6) In a geometrical question‖ in the third part of the book, the radius of a circle inscribed in a right-angled triangle, whose sides are 10 and 8, is given as '18 − ℞ von 164' (*i.e.* $18 - \sqrt{164}$).

(7) The expression 39₰ − $\frac{3}{512}$₰ occurs in a question¶ on alligation relating to coinage. The quantity $623\frac{29}{32}$₰ has to be divided by 16 and the result is given in this form. There seems no reason why it should be so given rather than in the form $38\frac{509}{512}$₰ (in which it appears in the edition of 1508). Possibly Widman obtained his result by writing the original quantity in the form 624₰ − $\frac{3}{32}$₰.

(8) Six persons have to divide 20 florins in the following proportions: the first is to have $1\frac{1}{2}$ fl + $\frac{1}{3}$, the second $2\frac{1}{2}$ fl + $\frac{1}{4}$, and the other four 1 fl** each. The solution is,

* Widman, p. 109′.

† *Id.*, p. 115. "Itm̄ eyner hat kaufft 6 eyer − 2₰ pro 4₰ + 1 ey". Borgi has a similar question (p. 115), "5. pome e vn danaro val 8 dn̄i men vn pomo. adimando che val el pomo". He solves it by the rule of false. More complicated questions of the same kind are given by Paciolo, pp. 105, 195. Those on the latter page are solved by algebra.

‡ *Id.*, p. 127′. If x be the interest for one year, then $(20+x)\left(1+\frac{x}{20}\right) = 30$, whence $x = \sqrt{600} - 20$.

§ *Id.*, p. 132. ‖ *Id.*, p. 217.

¶ *Id.*, p. 165. In the edition of 1500 it is printed 39₰ $\frac{3}{512}$₰.

** *Id.*, p. 195′. "Itm̄ 6 gesellen teylen 20 fl. Der erst sol haben $1\frac{1}{2}$ fl + $\frac{1}{3}$ Der ander $2\frac{1}{2}$ fl + $\frac{1}{4}$. Vnd die andern̄ 4 solln̄ gleich teil haben". The solution is "Machss also Reducir dye teyl facit $\frac{11}{6} + \frac{11}{4}$ die sūmir fa. $\frac{110}{24}$ addir die 4 gesellen dar zu facit $\frac{206}{24}$ ist $\frac{103}{12}$. . . "

The meaning is that the 20 fl are to be divided in these proportions. This mode of expression was general in Widman's time and continued for long afterwards, *e.g.* if 20 was to be distributed among three persons so that the first had $\frac{1}{2}$, the second $\frac{1}{3}$, and the third $\frac{1}{4}$ it was not meant that they were to receive these amounts, but merely that the 20 was to be divided in these proportions. The procedure was to take a number which contained the denominators, such as 12; the amounts then became 6, 4, 3, of which the sum was 13, and the amounts to be received were obtained as the fourth terms of the proportions 13 : 20 :: 6, 4, 3, and were therefore $9\frac{3}{13}$, $6\frac{2}{13}$, $4\frac{8}{13}$.

$\frac{11}{6}+\frac{11}{4}$ makes $\frac{110}{24}$, and adding 4 for the other four we have $\frac{206}{24}$, that is $\frac{103}{12}$, and "secz alsso

$$\begin{array}{cccc} & \frac{11}{6} & & 4\frac{28}{103}, \\ \frac{103}{12}-20 & \frac{11}{4} & \text{fact} & 6\frac{42}{103}, \\ & 1 & & 2\frac{34}{103}\text{''}. \end{array}$$

Here in each of the three cases + is used to add quantities of the same denomination and – is used to separate the first two terms in a proportion.

(9) Three persons have to divide 100 florins in the proportions, $\frac{1}{3}-\frac{1}{4}$, $\frac{1}{4}-\frac{1}{5}$, $\frac{1}{5}-\frac{1}{6}$. How much does each have?* The solution is to take a number which is divisible by 12, 20, 30: say 1800. Then $\frac{1}{12}$ of 1800 is 150, $\frac{1}{20}$ of 1800 is 90, and $\frac{1}{30}$ of 1800 is 60. The sum of these is 300. Then "secz alsso

$$\begin{array}{cccc} & 150 & & 50, \\ 300-100 & 90 & \text{facit} & 30, \\ & 60 & & 20\text{''}. \end{array}$$

Here in the fractions – connects quantities of the same denomination, and the subtraction can be at once effected: and in the proportions it is used to separate the first two terms.

Other uses of + *and* – *by Widman*, §§ 22–24.

§ 22. The – is also used to separate terms in a proportion in two other questions. In these it separates the third and fourth terms as well as the first and second, viz.

$$\begin{array}{lcl} & & 1910-32, \\ & 11630-198 & 4610-78, \\ & & 5110-89\dagger, \\ \text{and} & 353-305 & 140-120\frac{340}{353},\ddagger\ \text{\&c.} \end{array}$$

§ 23. In the rule of false the position and its error are placed in the same line, and the error has the + or – sign

* Widman, p 195′. "Itm̄ drey gesellen teylen̄ 100 fl vn̄ der erst sol habn̄ $\frac{1}{3}-\frac{1}{4}$ vnd der ander der $\frac{1}{4}-\frac{1}{5}$ Vnd der drit $\frac{1}{5}-\frac{1}{6}$". Here also $\frac{1}{3}-\frac{1}{4}$, &c., merely are the proportions in which the 100 fl are to be divided.

† *Id.*, p. 188. In the 1500 and 1508 editions the fractions are inserted, and the third and fourth terms are $1910-32\frac{602}{1163}$, $4610-78\frac{564}{1163}$, $5110-86\frac{1160}{1163}$. In the 1489 edition 4610 is misprinted 4910.

‡ *Id.*, p. 193′.

prefixed according as it is in excess or defect. Thus the position and its error are connected by the signs + and −. The examples* are

$$\begin{array}{ccc} 6+\frac{7}{8} & & 12-\frac{1}{20} \\ \times & \qquad & \times \\ 7+\frac{11}{16}† & & 13+\frac{9}{80}‡. \end{array}$$

§ 24. The mark − is used in another manner in a question relating to wool§, in which 60 lbs are bought at 55*Э* the lb, 50 lbs at 45*Э*, and 40 lbs at 35*Э*. In the solution of the question 60−55, 50−45, 40−35 occur, but the − here is used merely to connect each amount of wool with its cost per lb.

Some uses of + and − by Widman: possible misprints, §§ 25–26.

§ 25. As mentioned before (§ 9) the actual first appearance of either + or − in Widman's book is in the expression $\frac{1}{3}$ vn̄ $\frac{1}{4}+\frac{1}{5}$‖. The + may be merely a misprint; or, as De Morgan suggests, Widman may have been in the habit of using + in writing, and this + may have crept into print accidentally; or he may have used an abbreviation for 'and' in writing, which the printer expanded into vnd or vn̄, and which in this case he mistook for +.

The sign + occurs in the heading 'Regula augmenti + decrementi'¶. I think that the + is simply a misprint and

* The first question is in effect: A mark of one alloy contains 12 lots of silver, and a mark of another alloy contains 15 lots of silver (there are 16 lots in a mark). How are they to be combined so that a mark of the composition may contain 13 lots? If 6 lots of the first are taken and 10 of the second, the mark so found contains $13\frac{7}{8}$ lots of silver, and if 7 lots of the first and 9 of the second are taken, the mark contains $13\frac{11}{16}$ lots, whence the statement in the text which gives $10\frac{2}{3}$ of the first alloy and $5\frac{1}{3}$ of the second. In the second question, a person has bought ginger at 5 lb for the florin and pepper at 8 lb the florin. How many pounds of ginger and pepper, in equal quantities, have cost him 2 florins? Taking 12 lb (*i.e.* 6 lb of ginger and 6 lb of pepper) he finds that these have cost $1\frac{19}{20}$ fl, and taking 13 lb the cost is $2\frac{9}{80}$ fl, whence the statement in the text giving the result $12\frac{4}{13}$ lb.

† Widman, p. 201′. ‡ *Id.*, p. 202′. § *Id.*, p. 103′.

‖ *Id.*, p. 87. In the 1500 edition the + is omitted apparently by accident: it occurs in the 1508 edition.

¶ *Id.*, p. 112. The + occurs also in the editions of 1500 and 1508, but is replaced by ct in the Table of Contents at the end of these editions. De Morgan's comment is "The heading Regula augmenti + decrementi is probably a sort of joke, a use of + for logical aggregation: the regula augmenti alone is just before" (*l.c.* p. 207). Whatever its exact meaning, this comment shows that De Morgan regarded Widman's use of + as intentional. Tropfke clearly regarded the use of the + sign in '$\frac{1}{3}$ vn̄ $\frac{1}{4}+\frac{1}{5}$' and in 'augmenti + decrementi' as intentional, for he says that the former expression shows that Widman varies at will + and 'und', and the latter shows that he could use + when addition was not meant. "Es wechselt hier das Zeichen + beliebig mit dem wort 'und' . . . Ja das Pluszeichen erscheint auch da, wo gar keine addition vorliegt" (*Geschichte der elementar-mathematik*, vol. i., p. 131). [It may be noted that Tropfke quotes '9 fl $\frac{1}{3}$ und $\frac{1}{4}+\frac{1}{5}$ schilling' from Widman instead of '9 fl $\frac{1}{3}$ vn̄ $\frac{1}{4}+\frac{1}{5}$ eynss fl'.]

Widman's explanation of the 'Regula augmenti + decrementi' is "In dieser Regel soltu dich alszo halten Subtrahir die kleyner zal von der grossern̄ Vnd das

that the word should be et. Here again the abbreviation that Widman used for 'and' may have been mistaken for +. The − which occurs in $197 - \frac{16}{27}$ is clearly a misprint, the true value being $197\frac{16}{27}$*.

It is possible also that the + in $\frac{3}{5}+\frac{2}{5}$ and 20𝟿 + 15𝟿 in (1) and (2) of § 21 may be misprints.

§ 26. The first example of the 'Regula augmenti + decrementi' is: A man buys a sack of aniseed: if he pays 12𝟿 a pound he has 37𝟿 left: if he pays 18𝟿 a pound he is 44𝟿 short: how much does the sack weigh, and how much money has he? The solution is "subtrahir 12 von 15 bleyben 3 vnd das ist der teyler darnach addir + vnd − zeusam wirt 8", which gives 27 lbs as the weight of the sack. Here, as in

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vberige teyl. mit der minnerung vnd merung zusam geaddiret vñ der selbigen teylung quocient saget dyr zal...". It seems clear that the 'minnerung' and 'merung' are the 'decrementum' and 'augmentum', and as these are added together the use of + in the heading has some justification. The previous rule was the 'Regula Augmenti' which is thus described: "Subtrahir die kleyner anzal vonn der grossern vnd das vberig behalt zu deinē teyler. Darnach subtrahir auch dz kleyner residuū von dem grossern. vnd das vberig geteylt durch deynē vorbehalten teyler bericht die frag des gewichtes ader des gleichen...". The example is: if a man buys 9 lbs he has 13 gr left; if he buys 14 lbs he has 1 gr left: how much did 1 lb cost, and how many groschen had he? Here the 9 is subtracted from the 14 to give the divisor, and 1 is subtracted from 13 to give the dividend. The 'augmentum' is apparently the amount of money left over, called 'residuum' in the explanation. The next 'rule' after the Regula augmenti et decrementi is the 'Regula plurima', and the first question under it is: "Three articles cost as much more than 4𝟿 as four cost more than 10𝟿".

I do not know to what extent the numerous headings in the *Rechenung* were taken from previous writers, but it seems likely that some of them were due to Widman himself. They do not occur in Borgi or Paciolo. Questions of the same kind as those in the 'Regula augmenti + decrementi' are given on p. 104 of Paciolo, where they are included under the rule of false. They have no special heading.

The name 'Regula augmenti et diminutionis' was early assigned to the rule of false, for Leonardo Pisano in chapter 13 of his *Liber Abbaci* ('de regulis elchatayn, qualiter per ipsam fere omnes questiones abaci soluuntur') writes "Est enim alius modus elchataym; qui regula augmenti, et diminucionis appellatur, in quo ponuntur errores sub posicionibus suis" (*Scritti di Leonardo Pisano*, by B. Boncompagni, vol. i., Rome, 1857, p. 319); and an earlier Latin MS. on the rule of false printed by Libri in vol. i., pp. 304–376, of his *Histoire des sciences math. en Italie* has the title "Liber augmenti et diminutionis vocatus numeratio divinationis...quem Abraham compilavit..." (on this MS. see Cantor, vol. i, p. 627). It does not contain any question of the same kind as Widman's. Apianus, writing in 1527 (see 10° of § 42), begins his account of the rule of false "Dise regel wirdt von etlichen augmenti vñ decrementi auch zū zeiten Regula positionum genandt". The only other work in which I have found the headings 'Regula augmenti' and 'Regula augmenti et decrementi' with examples of the same kind as Widman's is the "Arithmetice Liliū Triplicis practice quam pulcherrimæ..." The copy I possess, which is the only one I know of, is incomplete, ending with the leaf E iii, and as there is no author's name on the title-page, and the colophon page is missing, I can only suggest Cologne as a probable place of printing, and *c.* 1510 as the date. The example under the first heading is: if a person buys 8 lbs of pepper he has 20 fl left, and if he buys 8 lbs he has 12 fl left: what is the cost of 1 lb? And the example under the second heading is: if a person pays 24 fl for each load of wine he has 52 fl left, and if he pays 30 fl for each load he is 80 fl short: how many loads did he buy and what was the cost of a load?

* Widman, p. 107. The misprint is repeated in the editions of 1500 and 1508.

the question relating to figs (§ 11), + and − mean the terms which have these signs, and the use + and − in this sense is noticeable as the signs do not occur in the statement of the question. To obtain the amount of money Widman multiplies 12 by 27 and adds 37, or multiplies 15 by 27 and subtracts 44.*

In the next example if a man pays his workmen 59 each he has 119 over and if he pays 99 he is 179 short.

In the question of the eggs (No. 3 of § 21) the use of the signs is reversed, *i.e.* they occur in the question, but not in the solution. This example is "Itm̄ eyner hat kaufft 6 Eyer − 29 pro 49 + 1 ey", and the solution is "Addir dy gemyndertn̄ 29 zu 49 werdn̄ 69 vnd dz ist der zeler. vnd darnach Addir auch die kleyner der eyer gemyndert zu der grossen irn̄ gleichn̄ Ader subtrahir das kleynst gemert von der grossern ezal irss gleichn̄ als 1 ey von 6 pleybn̄ 5 vnd ist der nenner des vorgefundnen zelerss. vnd stet also $\frac{6}{5}$ vnd so tewer kumpt 1 ey". Here 'gemert' and 'gemyndert' seem to represent + and − and the rules are equivalent to "Add −1 to 6 or subtract +1 from 6".†

A distribution question of Widman's, §§27–28.

§ 27. Another example‡ of Widman's should be noticed because he uses the word *mer* when we might have expected +. The question is to divide 384fl among 4 persons so that the first is to have $\frac{2}{3}$ and 6 more ($\frac{2}{3}$ vnnd 6 mer), the second $\frac{3}{5}$ and 8 more, the third $\frac{5}{6}$ and 10 more, the fourth $\frac{7}{8}$ and 6 more. Widman's solution is to find§ a number which contains all the denominators of the fractions; he takes 360, of which $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{6}$, $\frac{7}{8}$ are 240, 216, 300, 315: to these he adds 6, 8, 10, 6 respectively, giving 246, 224, 310, 321, their sum being 1101. He then forms proportions, which he writes as follows:

	$\frac{2}{3}$ 6 mer		246		85 fl $\frac{293}{367}$,
	$\frac{3}{5}$ 8 mer		224		78 fl $\frac{46}{367}$,
1101		Als		Facit	
	$\frac{5}{6}$ 10 mer		310		108 fl $\frac{44}{367}$,
	$\frac{7}{8}$ 6 mer		321		111 fl $\frac{351}{367}$.

* The general rule for finding the amount of money when the weight of the sack has been found is "weliche zal [the number of lbs] szo sy gemultiplicirt wirt mit der kleynern anzal vn̄ die grosser mynnerung von dem product subtrahirt wirt Ader widerumb. das darnach vberpleybet bericht die ander frag". There is some confusion in this rule, for the number of lbs should be multiplied by the greater number if the 'mynnerung' is to be subtracted. In the example the rule is applied correctly.

† Widman, p. 115. This rule, which has the heading 'Regula pulchra', follows immediately the 'Regula plurima' mentioned in the second note to § 25.

‡ *Id.*, p. 196′.

§ "Find eyn zal dar yn du die gebrochen alle habst Vn̄ ist 360 Nym $\frac{2}{3}$ von 360 ist 240 vnd $\frac{3}{5}$ von 360 ist 216...".

This solution implies that Widman regarded the shares of the four persons as proportional to $\frac{2}{3}\lambda+6$, $\frac{3}{5}\lambda+8$, $\frac{5}{6}\lambda+10$, $\frac{7}{8}\lambda+6$, λ being an arbitrary number which may be chosen at convenience, and he takes 360, but of course he might as well have taken 120, 240 . . . or indeed any other number*. Thus the question as Widman interprets it is indeterminate. To render it determinate we must suppose that after the persons have received 6, 8, 10, 6 respectively, the residue (*i.e.* 354 fl) is to be divided among them in the proportions of $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{6}$, $\frac{7}{8}$. In this case the amounts which they receive are $85\frac{39}{119}$ fl, $79\frac{47}{119}$ fl, $109\frac{19}{119}$ fl, $110\frac{14}{119}$ fl.†

§ 28. It is interesting that Widman should have written mer instead of using +, for I do not think he would have scrupled to write $\frac{2}{3}+6$, meaning $\frac{2}{3}$ of some quantity + 6 fl. It is likely that he took the question from some earlier writer, and left it unchanged. If he had examined the question, he would have remarked the ambiguity.‡

Widman's general use of + *and* −: *criticism of De Morgan's 'warehouse' theory*, §§ 29–31.

§ 29. I think these examples show that Widman used the signs + and − in all the ways in which they are used in algebra, that is to say, they connect by addition and subtraction

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* Putting $\lambda = 120k$, the portions received by the four persons are

$$\frac{128\,(80k+6)}{119k+30},\quad \frac{128\,(72k+8)}{119k+30},\quad \frac{128\,(100k+10)}{119k+30},\quad \frac{128\,(105k+6)}{119k+30}.$$

Widman's solution corresponds to $k=3$.

† Paciolo gives questions of this kind and solves them correctly (*i.e.* he so interprets them that they have but one solution): he connects the fraction and the absolute number by p̄ (piu), and he also gives questions in which they are connected by m̄ (meno). Thus No. 4 on p. 150 is "Doi guadagnā 100. al p°. tocca la mita. p̄. 5. al secōdo. el. $\frac{1}{3}$. p̄. 4."; the question is equivalent to dividing 91 in the proportion of $\frac{1}{2}$ to $\frac{1}{3}$, *i.e.* one has $\frac{3}{5}$, and the other $\frac{2}{5}$, of 91. The next question is to divide 100 "el p°. die hauer la mita. p̄. 3. el secōdo el. $\frac{1}{3}$m̄. 5". Here 102 is to be divided in the proportion of 3 to 2.

In the *Arithmetice Lilium* (mentioned on p. 17 in the second note to § 25), in which the author seems to follow Widman closely, there is a similar example which is solved in the same manner. The question is to divide 100 fl among three persons so that the first has $\frac{1}{2}$ and 3 fl ('$\frac{1}{2}$ et 3 aureos'), the second $\frac{1}{3}$ and 2 fl, and the third $\frac{1}{5}$ and 4 fl. A number containing all the denominators is taken, viz. 30, and, the three shares being then represented by 18, 12, 10, their values are obtained as the fourth proportionals to 40, 100, and 18, 12, 10, and are found to be 45, 30, 25. The heading of the example in the *Lilium* is 'Regula Divisionis', Widman's example being placed under 'Teylung'. The first satisfactory solution of such a question which I have noted is in Rudolff's *Kuntslicke Rechnung* (8° of § 42), where the shares are '$\frac{1}{2}$ vnnd 6 fl', '$\frac{1}{3}$ vnd 1 fl', and '$\frac{1}{4}$ minus 2 fl'. See §§ 51, 61.

‡ Drobisch, referring to Widman's Geometry, wrote "Iam vero luculenter apparet, in his Widmannum nostrum tantummodo compilatorum fuisse. Nam clausis quasi oculis falsis agrimensorum Romanorum regulis utitur, quanquam ipse in antecedentibus alia, eaque, certe ex parte, rectiora docuerit" (*De... Widmanni...compendio*, p. 32). This remark seems to me to apply also to some of the regulæ and problems in the Arithmetic.

quantities of different denominations and kinds such as centners and lbs, shillings and heller, eggs and pence, rational and irrational quantities, and quantities of the same kind, such as $\frac{1}{3}$ and $\frac{1}{4}$, $1\frac{1}{2}$ fl and $\frac{1}{3}$ fl, &c. They are also used, in the rule of false, to denote excess and deficiency *.

De Morgan wrote †: "In favour of the warehouse theory it may be added that Widman is very chary of the use of the signs + and − throughout the part of his work which treats of fancy problems or of the higher rules of commerce: it may be that he was little accustomed to the signs except in the class of problems in which they reached him as data". Gerhardt stated that they occurred in an isolated manner and not as having a general application, and Eugene Smith (probably following De Morgan) says that they "are not used, however, as signs of operation, but as symbols of excess or deficiency in warehouse measures".‡

§ 30. I disagree with all these comments.§ The natural inference from the book seems to me to be that Widman was accustomed to denote plus and minus by + and − in algebra, and that when he came to write a mercantile arithmetic he used the signs he was familiar with. I should also have thought it likely that the signs were already known, but this does not appear to have been the case ‖. Whether

* Stifel in the *Arithmetica Integra* (1544) describes the use of the signs + and − as primarily to connect quantities, the proportions of which could not be exactly given (such as irrationals), and quantities of which the proportion is unknown (as when one is an unknown), but adds that besides this double necessity they are used whenever it is convenient ("Primo enim utimur illis in numeris talibus, quorum proportio præcise dari non potest: ut in numeris irrationalibus. Secundo utimur eis in numeris talibus, quorum proportio ignota est, & si præcisa sit: ut in numeris cossicis, dum numeros quærimus nobis absconditos. Sed praeter necessitatē hanc duplicem, utimur eis commoditatis gratia, ut aliquid per ea monstremus aut doceamus", p. 248'). These words seem to me to express accurately the uses of + and −. They are essential when the quantities are to be added, or subtracted the one from the other, and the addition and subtraction cannot be performed, and it is convenient to use them in many other cases.

† *loc. cit.*, p. 209.

‡ *Rara Arithmetica*, p. 39.

§ Tropfke takes the same view as I do (and is therefore opposed to those of De Morgan and Gerhardt) with regard to Widman's general use of the signs, for he says that though their employment in the fig question gives the impression of mercantile practice, still his collective treatment of them enables us to recognise "his skill in calculating with the signs, the free use of which in other different problems shows that to him + was not merely a word nor was it, in connection with −, merely a mercantile mark, but that both signs were already to him genuine symbols". ("Die Verwendung der Zeichen + und − macht in den hier aufgestellten Rechnungen den Eindruck, als ob sie aus der kaufmännischen Praxis hervorgegangen seien. . . . Das Zusammenziehen einer ganzen Reihe solcher Ausdrücke lässt die Geschicklichkeit Widmann's erkennen, mit diesen Zeichen zu rechnen; der freie Gebrauch in verschiedenen anderen Aufgaben verrät geradezu, dass ihm das + nicht mehr einen Wortersatz, oder, in Gemeinschaft mit dem −, nur eine kaufmännische Signatur darstellt, sondern beide Zeichen ihm bereits wirkliche Symbole geworden sind". *Geschichte*, vol. i., p. 131).

‖ See, however, Part III. of this paper.

Widman derived + and – from the warehouse or from some other source, he used them freely, and it seems to me not unlikely that some of his questions may have been specially devised to exhibit their use.

§ 31. The 'warehouse theory' of the origin of the signs + and – may have two distinct meanings: it may mean (i) that the warehousemen used these signs and the arithmeticians (or algebraists) did not, so that in writing a mercantile arithmetic Widman used the signs that his readers would understand, and that the warehouse signs thus passed into arithmetic; or (ii) that Widman, desiring to replace plus and minus, or more and less, by symbols, derived + and – from the warehouse and not from abbreviations in writing.

The first view is clearly that which was held by De Morgan.* The points in its favour are (i) that the signs first appear in a commercial arithmetic, the title of Widman's book being "Behēde vnd hubsche Rechenung auff allen kauffmanschafft", and (ii) that their meaning is explained very briefly, and that they first occur in the statement of the weights (§ 11).

The second point seems unimportant, as the interval between the use of the signs in the weights and the explanation of their meaning is so slight that they may be said to be defined as soon as they are introduced; and in any case such irregularities are not uncommon in all early books†.

* In a letter (on the history of + and –) to *The Athenæum* (published in the number for Oct. 29, 1864, p. 565) De Morgan explained the warehouse theory even more fully than in his paper quoted in § 12: "Suppose a warehouse in which bales are frequently weighed which are usually something over or under 300 pounds. Three weights would be put into the scale of 100 pounds each; and each bale would then require more weights in one scale or the other. Thus a bale of 325 pounds would take 25 pounds in the scale of the weights; one of 288 pounds would take 12 pounds in the scale of the goods. In weighing bale after bale, and making entries, it is probable that the whole result would not be formed at once, but, '25 *more*', or, '12 *less*', would be entered in a warehouse-book, or chalked or painted on the bales themselves. Some signs might be used instead of *more* or *less*: and + for *more* and – for *less* might suggest themselves. In this case, when the entries came to be fully made, 300 + 25 and 300 – 12 might easily take their place in a column". He then says that this is not conjecture, "for I have found in an old work of commercial arithmetic, which I shall have occasion to describe when I collect my notes on the subject, a question proposed and solved, which shows that the preceding use of + and – was made before 1489, and that it was not thought necessary to explain to the reader what + and – stood for. The explanation above given is certainly implied in the question and solution, and its absence shows that the question would be understood without it by the readers for whom the book was intended". De Morgan is not correct in saying that Widman does not explain what + and – stand for.

† Drobisch thought that Widman's mode of using + and – showed that they were sufficiently well known (§ 13), but the fact that their meanings are explained affords some evidence to the contrary. Even when a word or sign could clearly not have been known, Widman does not feel obliged to explain it, for in the geometrical part of this book he uses the word *cossa* and its cossic symbol without any explanation (see § 34); nor does he explain the meaning of the connecting lines × and = in the rule of three.

But the manner in which the signs first occur certainly gives the impression that they are not complete novelties, and suggests that they may have been employed as marks of excess and defect by warehousemen in the manner described by De Morgan. Widman being aware of this may have adopted them as convenient symbols for plus and minus, but, whether this was the case or whether he derived them from other sources, we should expect him as an algebraist to use them as plus and minus are used in algebra. This he does, and he seems to me to have even done more, and to have invented questions to show the use of the signs and how they are to be treated. Thus in the question about pepper which follows immediately after the question about figs, in which centners and pounds are connected by the signs + and −, he connects shillings and heller by the sign −. I do not think such an example would have arisen naturally, and it seems not unlikely that Widman may have devised this question and some others in order to illustrate the use of the signs and the manner of treating minus quantities.

I cannot agree with De Morgan's remark that Widman may have been "little accustomed to the signs except in the class of problems in which they reached him as data", for they could not have been data in questions relating to money, and how could the prices $6+\frac{1}{8}$, $5+\frac{1}{8}$, $4+\frac{1}{8}$, $3+\frac{1}{8}$, $3-\frac{3}{4}$, $2+\frac{1}{8}$ (in florins) have arisen as data? The questions which De Morgan describes as 'fancy problems' were mainly those that were taken from earlier writings, as several of them occur in the early Italian arithmetics*.

The use of the cossic notation by Widman, §§ 32–37.

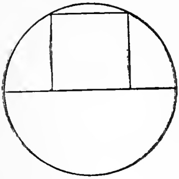
§ 32. That Widman was an algebraist is shown by his own book, for in the third part, which relates to geometry, he uses the words cosa and census and the cossic sign for cosa. They occur in the solution of the problem of the inscription of a square in a semicircle.

The problems and solutions are as follows:†

"Itm̄ wiltu aber in eynen halben cirkel machen eyn quadrat auff das grost. vn̄ wilt wissen wie lang der seyten eȳne sey Nu secz also oben der diameter des halben cirkelss sey 12. vnd eȳnn lini die von oben herab perpencularitcr gezogen wert sey 6 Darnach machss also multiplicir den diametrū in sich selbst wirt 144 vn̄ teylss durch 5 kūmen $28\frac{4}{5}$ vnd ℞ von $28\frac{4}{5}$ ist das quadrat durch die seyten.

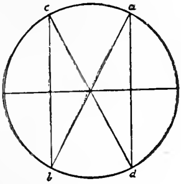
* Some of these questions are referred to in Part III.

† Widman, p. 217'.



"Het ich aber also gesprochen Es ist eyn halb cirkel dess corda von dem arco ist 12 vn̄ sagitta ist 6 dar eyn wil ich machn̄ das grost quadrat so ich mag Nu ist die frag wie wil der bald cirkel sey machss also Nu du sichst das die halb rotund gleich ist dem quadrat vnd ich wil eynn andere rotund hin eyn machn̄ vnnd wil die selbige rotund zueygen der selbn̄ halben vnd thu ym also Ich mach eyn rotund hyn eyn v̄ dar eyn mach ich zwē quadrat also das eyner zwir so lāgk sey als weyt als in dieser figur.

"Vnd darumb soltū seczē dz vom a piss zum c sey eyn cossa. vnd vom c zum b auch eȳ cossa. vnd also sprich dz eyn quadrat weyt sey 1 cossa vnnd langk 2 cossa Darnach wart wie gross eyn quadrat sey das do weyt sey 1 cossa. Vnd zweyer langk. mltiplicir 1♃. durch 1♃ wirt 1 zēsꝰ vū [vn̄] multiplicir 2♃. durch 2♃. werdē 4 zēsꝰ addirss zusammen wern̄ 5♃. das ist vom a zum b vnd auch vō c zum d auff das geneust. Vn̄ ist oben berurt das der dyameter der rotūd sey 12 darumb quadrir 12 werden 144 dz teyl durch zēsꝰ als 5 so kūmē $28\frac{4}{5}$ und szo vil ist die ra. das ist ra. von $28\frac{4}{5}$ vnnd ist geseczt das. dz quadrat sey auff yder seyten 1♃ Darumb ist eyn seyten ℞ von $28\frac{4}{5}$ und also hastu das vbereȳ kumpt ♃ mit der andern regel".



The cossic symbol for res or cosa, which is here denoted by ♃, is the abbreviation which was used in manuscripts and early printed books to denote the termination *rum*.

§ 33. Widman first gives the rule, viz. that if the diameter of the semicircle is 12, then the side of the square is the square root of $28\frac{4}{5}$. He then proceeds to prove it. He completes the circle by adding an equal semicircle below, and produces a vertical side of the square to meet the lower semicircle in *b*. His words are: "I make a circle and therein I make two squares so that one is twice as long as wide as in the figure". The meaning clearly is that the length (which is the side of the larger square) is twice the width *ac* (which is the side of

the smaller square). He then proceeds, using cossic words and a cossic symbol:

"You must suppose that from a to c is 1 cossa, and from c to b is also 1 cossa, and say that a square is 1 cossa wide and 2 cossa long. How large will a square be that is 1 cossa wide and [how large will one be] that is 2 cossa long? Multiply 1 ♃ by 1 ♃, which is 1 zensus, and multiply 2 ♃ by 2 ♃, which is 4 zensus, add together, they are 5 ♃ [obviously an error for 5 zensus], that is from a to b and also from c to d. exactly. And it is stated above that the diameter of the circle is 12, square 12, it becomes 144, divide by zensus which is 5 it is $28\frac{4}{5}$ and so much is the root: that is the root of $28\frac{4}{5}$, and it has been supposed that the square is on the side 1 ♃, therefore the side is ℞ of $28\frac{4}{5}$, which is in accordance with the above rule".

The process is quite clear, ac is 1 ♃, cb is 2 ♃, acb is a right angle as ab is obviously a diameter, and therefore the squares on 1 ♃ and 2 ♃ are equal to 144, therefore 5 zensus is equal to 144, and therefore 1 zensus is $28\frac{4}{5}$. The statement that from c to b is 1 cossa is clearly a slip: it should be 2 cossa. A paraphrase of the sentence would be "form a square whose side is 1 cossa wide, and one whose side is 2 cossa long". Widman uses the word 'wide' when in the figure the side is horizontal and 'long' when it is vertical, *i.e.* as denoting the base and cathetus of a right-angled triangle. When Widman says 'divide by the zensus', meaning 'divide by the coefficient of the zensus', he is following the practice of the time and of long afterwards, when the direction to divide by a term means that its numerical coefficient is to be the divisor.

§ 34. This solution establishes beyond question that Widman was an algebraist, and was conversant with algebra when he wrote his *Rechenung** (although, as will be seen, there is abundant independent evidence upon this point). It is also

* The manner in which Widman uses the signs + and − in the arithmetical portion of the *Rechenung* seems throughout to show algebraical influence, so that it is satisfactory to have the cossic notation actually used in the book. It is clear that Widman's intention was to avoid algebra in his arithmetic, for in the dedication he says that the old masters of the art have let themselves go astray in giving perplexing and troublesome rules such as those of algebra or the Coss, the Data [of Jordanus], the rules of proportion, &c., which are difficult, wearisome, and incomprehensible to common people. ("Du hast betracht ynn deinem gemute. Dass die aldē meyster der kunst der Rechnūg Irenn nach komendē schwere Regelñ tzunornemen vñ muesam tzuuerfuren gelassen haben Alss do seynn die Regel Algobre ader Cosse genant dass buch. Data genant vñ die Regel proportionū vnd ander der gleychen. Die do alle dē gemeynem volck tzu schwer verdrossenn vnd vnbegreyfflich seyn". p. 2).

He refers also to the rule of Coss in introducing the rule of false, his words being "Nu soltu wissen das Regula falsi ist eyn Regel durch weliche man aller Regal frag (hind an gesaczt Regula Cosse) machñ mag" (p. 200).

valuable as an example of Widman's style, for he gives no explanation of cossa, census, or his cossic symbol, and though these might be known to an algebraist, they would not be known to the readers for whom his book was intended. Even if the signs + and – were Widman's own invention I do not think it would have occurred to him to explain them more fully than he has done in the first question where he uses them.

§ 35. I have thought it worth while to reproduce the whole of Widman's treatment of this problem, for it is of great interest as being probably the first occurrence of algebra in any printed book. The fact that Widman used algebra in the *Rechenung* has indeed been mentioned by Drobisch and others, as will now be seen; but it is surprising that so little attention has been paid to this first appearance of the cossic notation in print.

Drobisch, in giving an account of Widman's geometrical problems, says that he has not seen in any previous author the inscription of an equilateral triangle, or of a square, in a semicircle; and of the latter he remarks, "Et alterum quidem cossicæ artis ope resolvere studet, sed rem perobscure, ne dicam confuse tractat. Quod voluit hoc est". He then explains how by taking a side of a square to be ♃, a right-angled triangle can be constructed having its other side 2♃ and the diameter of the circle for its hypothenuse, the square of which is therefore 5 census, whence ♃ is equal to $\sqrt{28\frac{1}{5}}$.*

De Morgan makes no reference to Widman's use of algebra.†

Gerhardt only refers incidentally to the use of algebra. In his account of the geometrical problems in the *Rechenung* he mentions the inscription in a semicircle of an equilateral triangle and the largest possible square, 'die letztere auch mit Hülfe der Algebra'‡.

Treutlein merely mentions that the *Rechenung* is the first printed work in which the sign for cosa occurs, and he reproduces the sign as used by Widman.§

Tropfke prints the entire paragraph "Und darumb . . . regel" (quoted in § 32) which contains the solution, merely

* *De...Widmanni...compendio*, p. 32.

† De Morgan expresses surprise that those who noted the chain rule in Widman made no mention of + and –, but he himself passes over without remark the use of the cossic sign and words.

‡ *Geschichte*, vol. i, p. 34.

§ *Zeitschrift für Math. und Phys.*, vol. xxiv. (Supp.), p. 32: "Das die Vorzeichen + und – schon früh benützende Buch von Widman (1489) ist das erste gedruckte Schriftwerk, bei dem jenes Zeichen (und zwar in der Form ♃) in der Bedeutung für *cosa* vorkommt".

introducing it with the words "Die einzige Stelle, an der Widmann algebraisch rechnet, findet sich S. 215^b-216^b"* In his table† of the forms of the cossic signs used by early writers he gives Widman's sign for cosa and zēs⁹ for census, but the latter cannot be regarded as a sign, for it is the full word zensus merely abbreviated by the contractions usual in writing and printing.

Cantor does not seem to mention the use of algebra by Widman.

§ 36. Although Treutlein and Tropfke reproduce Widman's symbol for cosa, I do not think that it has been explicitly noted that it is the contraction in general use for the termination *rum*. This suggests that it may not be an exact reproduction of the symbol actually used by Widman in writing, but merely the nearest symbol to it which the printer had among his type.

Drobisch states that he has not found the inscription of a square in a semicircle in any earlier work. There can however, I think, be no doubt that it is an old problem, for it occurs in Paciolo on p. 53′ of the second part (geometry) of the *Summa*, where it is briefly and effectively treated by algebra.‡ It is remarkable that Widman should have given the algebraical solution of a geometrical problem, while in the arithmetic he was generally content merely to state a rule and give the result, as *e.g.* in the question mentioned in (4) of §21, where the result could only have been obtained by the solution of a quadratic equation.

A study of the *Rechenung* shows that Widman was much more an arithmetician and algebraist, than a geometrician, and it seems not unlikely that his geometrical propositions were taken from some manuscript without much examination.

§ 37. It will have been noticed that in the solution of the problem in § 32 both ra. and ℞ are used to express the square root. In other places also in the geometry ℞ is used for square root, *e.g.* on pp. 214 and 216, where '℞ von $36\frac{3}{4}$' and '18 − ℞ von 164' occur.§

* *Geschichte*, vol. i., p. 317. Tropfke's pp. 215^b-216^b are on my system of paging 217′–218′.

† *Id.*, p. 191.

‡ "Di Io fo positiō fia larga. 1ª. cosa fia. 2. cose. Doue ilsuo diametro fia la ℞. di. 5. cose. Adōca ℞. di. 5. cose sōno iguali a. 12. Onde la cosa varra la. ℞. di. $28\frac{4}{5}$".

§ In the question on interest in the arithmetic, referred to in (4) of § 21, Widman writes 'die wurzel von 600–20' for $\sqrt{600-20}$.

The sign for cosa, which I have denoted by ♃, occurs again on p. 231, in the expression '2♃. Von 725', but it is evident that here 2♃ is a misprint for ℞.*

Part II.

Introduction, §§ 38–39.

§ 38. We know that Widman uses the signs + and – in his statement of the fig question to connect centners and lbs, and that in the pepper question which immediately follows it he connects denominations of money by the sign –. The signs + and – thus make their first appearance in print in the commercial questions of an arithmetic, and apart even from the use of the signs it is of some importance to examine to what extent this class of questions was repeated in succeeding arithmetics, and to endeavour to decide whether they represented actual transactions or were merely invented as exercises in arithmetical operations involving the use of the signs + and –.

§ 39. I have not seen the Bamberg Arithmetic (1483), but from the accounts of it given by Unger† and Cantor‡ it does not seen likely that it contains any question of this type (*i.e.* in which some of the data are expressed as one amount diminished by another). I have found no such question in Borgi (1484) or Paciolo (1494), so that it is quite possible that this kind of question originated in Widman's desire to exhibit the uses of + and –.

I have examined with some care the principal arithmetics and algebras published in Germany after Widman's *Rechenung* up to 1550 in order to determine to what extent + and – were used by Widman's successors, and whether questions of the type of Widman's fig and pepper questions were repeated, and if so how they were expressed. Some other matters of interest, such as the use of plus and minus, mer and weniger, tara and fusti, &c., which were suggested by the examination of these books, are also included.

List of sixteen books on arithmetic or algebra subsequent to Widman's, § 40.

§ 40. The following list includes the chief practical or commercial arithmetics, and also the algebras, published in

* The question is equivalent to finding the hypothenuse of a triangle whose base is 25 and cathetus 10, and the answer is $\sqrt{725}$.

† *Die Methodik*, pp 37–40.

‡ *Vorlesungen*, vol. ii. (second edition), pp. 221–227.

Germany subsequent to Widman's *Rechenung* of 1489 up to the middle of the following century:

1°. Huswirt (Cologne, 1501).
2°. Köbel (Augsburg, 1514).
3°. Böschensteyn (Augsburg, 1514).
4°. Grammateus (Vienna, 1518: arithmetic and algebra: this being the first German algebra).
5°. Grammateus (Nuremberg, 1521).
6°. Riese (Erfurt, 1525).
7°. Rudolff (Strassburg, 1525: algebra preceded by some arithmetic).
8°. Rudolff (Vienna, 1526: arithmetic).
9°. Peer (Vienna, 1526).
10°. Apianus (Ingoldstadt, 1527).
11°. Albert (Wittemberg, 1541).
12°. Stifel (Arithmetica integra, Nuremberg, 1544).
13°. Stifel (Deutsche Arithmetica, Nuremberg, 1545).
14°. Spenlin (Augsburg, 1546).
15°. Riese (Leipzig, 1550).
16°. Stifel's edition of Rudolff's Coss (Königsberg, 1553: algebra preceded by some arithmetic).

Unless otherwise stated all these books are arithmetics, and, except 12°, are written in the German language.

Titles of the sixteen books, §§ 41–42.

§ 41. I now give the titles of these sixteen books, and in all cases the title given is that of the actual edition which I have used myself: and when this is not the first edition it is so stated, and the date of the first edition is given. In the case of eight of the books, viz. 1° (Huswirt), 2° (Köbel), 3° (Böschensteyn), 10° (Apianus), 12° (Stifel), 13° (Stifel), 15° (Riese), 16° (Stifel's Rudolff), the title-page is reproduced in facsimile in Eugene Smith's *Rara Arithmetica.* For these books therefore I have given only an abbreviated title and a reference to *Rara Arithmetica.* In the case of 4° (Grammateus), 5° (Grammateus), 6° (Riese), 8° (Rudolff), 11° (Albert), only later editions are described in *Rara Arithmetica*, while 7° (Rudolff) and 9° (Peer) are only referred to; and 14° (the first edition of Spenlin) is not mentioned. The titles of these books are therefore given at greater length or in their entirety.

A great deal of information with respect to other editions of the sixteen books is given in *Rara Arithmetica*, which also contains notices of some other works, such as Reisch's *Margarita Philosophica* (1503), and the various *Algorithmi*, which I have omitted as containing only a trifling amount of commercial arithmetic. A general account of several of the arithmetics included in the above list, with bibliographical information, is contained in Unger's *Die Methodik der praktischen arithmetik* (Leipzig, 1888).

§ 42. The following are the titles of the books:*

1°. [Johannes Huswirt] "Enchiridion nouus Algorismi summopere visus..." (Cologne, 1501). A facsimile of the title-page is given on p. 75 of *R.A.*†

2°. [Jakob Köbel] "Ain New geordnet Rechen biechlin auf den linien mit Rechenpfeningen..." (Augsburg, 1514). A facsimile of the title-page is given on p. 103 of *R.A.*‡

3°. [Johann Böschensteyn] "Ain New geordnet Rechen biechlin mit den zyffern den angenden schülern zü nutz... durch Ioann Böschensteyn von Esslingen..." (Augsburg, 1514). A facsimile of the title-page is given on p. 101 of *R.A.*

4°. [Henricus Grammateus] "Ayn new kunstlich Buech welches gar gewiss vnd behend lernet nach der gemainen regel Detre, welschen practic, regeln falsi vñ etlichē regeln

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* *Rara Arithmetica* is so often mentioned in this section that it will be referred to by its initials *R. A.*

† Besides the editions mentioned by Eugene Smith in *R. A.* (p. 74) there is one of 1524 bearing the title "Enchiridion artis nvmerandi, paruo admodum negotio omnem calculi praxim docens in integris, minutijs ualgaribus, & proiectilibus: regulis aliquot mercatorū additis nequaquam contemnendis. Apud sanctam Vbiorum Agrippinā in ædibus Eucharij Ceruicorni, Anno M.D. xxiiii". At the end there is "Enchiridij Iohannis Husvuirt Sanensis de arte Calculatoria, Finis": followed by the colophon "Impensis integerrimi bibliopolæ M. Gothardi Hittorpij, ciuis Coloniensis."

‡ An edition with a similar title and having the same number of leaves was published at Oppenheim in the same year (1514). This edition is referred to in *R. A.* (pp. 102, 106), and its title is quoted by Unger (p. 44). A number of other editions of Köbel's works are mentioned in *R. A.* (pp. 100–113); but an edition printed at Oppenheim in 1517, which is not referred to in *R. A*, differs so essentially from that of 1518 (also printed at Oppenheim), the title-page of which is given in facsimile in *R. A.*, that it seems worth while to quote it: "Eyn Neüw Rechēbüchlein. Vff den Linien vñ Spaciē, Mit den Rechenpfennigē: ytzo: zü dez Zweyten male, Mit viln Züsetzen, güten Leren, vnnd Exempeln, Zü Oppenheym Geordent Vnd Getrückt. . . .", then eight verses beginning "Pythagoras Der sagt für war", and a diagram of a Rechenbanck with three 'Bankiers'. The colophon is "Der Zweyt Truck zü Oppenheym, mit vil züsetzen, Geordent vnd Getrückt. MCCCCC.XVII". The book is in the British Museum Library. It may be mentioned that there are 24 verses altogether, as they run over the verso of the title-page, there being eight on the title-page and 16, beginning "In Zal in Masz vn in Gewicht", on the verso. The 24 verses differ but slightly from those which Unger quotes from Adam Riese (1529) on p. 63 of *Die Methodik*. The ten verses on the title-page of the 1518 edition (reproduced in *R. A*) are quite different.

Cosse mancherlay schöne vñ zuwissen notürfftig rechnũg auff kauffmanschafft. Auch nach den proportion der kunst des gesanngs jm diatonischen geschlecht auss zutaylē monochordũ, orgelpfeyffē vñ ander instrument auss der erfindung Pythagore. Weytter ist hierjnnen begriffen buechhalten durch das Zornal, Kaps, vnd schuldbüch Visier zumachen durch den quadrat vnnd triangel mit vil andern lustigen stücken der Geometrey. Gemacht auff der löblichen hoenschül zü Wieñ in Osterreich durch Henricũ Grammateum, oder schreyber von Erffurdt der siebē freyen künsten Maister". The preceding is red: then in black "Mit Kayserlichē gnaden vnd Priuilegien das buech nicht nach zu truckē in sechs jarē" (Nuremberg, 1521?). The colophon is "Gedruckt zu Nürnberg durch Iohannem Stüchs für Lucas Alantsee Büchfurer vnd Bürger zu Wien". The first edition was printed in 1518 at Vienna: and the full title, which is long and resembles that of the present edition, is given in full by Unger (p. 47).*

5°. [Henricus Grammateus]. "Behend vnnd khunstlich Rechnung nach der Regel vnd welhisch practic, mit sambt züberaittung der Visier ym quadrat vnd triangel. Gemacht auff der loblichē hohē schul zu wienn durch maister Henrichen Grammateũ". The colophon is "Gedruckt vnd volendet zu Nürnberg durch Iohannem Stüchs ym iar nach Christi geburt. M.D. XXI". This book is referred to in *R.A.* (p. 123), where it is stated that it is an extract from the work of 1518. A considerable portion is the same as in the larger work, but not the whole. This book is not referred to by Unger.

6°. [Adam Riese]. "Rechenung auff der linihen vnd federn in zal, masz, vnd gewicht auff allerley handierung, gemacht vnd zusamen gelesen durch Adam Riesen von Staffelstein Rechenmeyster zu Erffurdt im. 1522. Iar. Itzt vff sant Annabergk, durchin fleissig vbersehen, vnd alle gebrechen eygent-

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* I have not seen the first edition (Vienna, 1518), but I do not doubt that the edition, of which the title is given in the text, is an exact reprint. This edition, which is in the British Museum Library, is by the same printer at Nuremberg as 5° (Grammateus' *Rechnung*). I have given the long title in full because it differs from that of the 1518 edition, which (except for trivial verbal differences) was followed in those of Frankfort, 1535 and 1572. The full title of the 1518 edition is given by Unger (p. 47), and the title-page of the 1535 edition is reproduced in facsimile in *R.A.* (p. 124).

The date of the Nuremberg edition described in the text is probably 1521. This I infer from the fact that the blank pages headed 'Zornal,' 'Kaps,' and 'Schuldt Buech' have the heading 1521, which is therefore likely to be the year of publication. It is also the date of the abridgment, 5°. The date of the privilege is July 20, 1518. The only other edition that I have seen myself is that of 1572.

There is less on the page in this 1521 edition than in that of 1518, for Treutlein (*Zeitschr. für Math, u. Phys.*, vol. xxiv., Supp. p. 13) describes the latter as containing 94 leaves, while the edition of 1521 contains 124 leaves (including the title-page leaf and colophon leaf).

lich gerechtfertiget, vnd zum letzten eine hübsche vnderrichtung angehengt." (Erfurt, 1525). The colophon is "Gedruckt vnnd volendet zu Erffordt durch Mathes Maler zū schwartzen Horn am abent Nicolay ym Iar 1525". The preceding page (K iii′) concludes with the words "Datum vff sandt Annabergk Dinstag nach Martini. Im 1525". There was an earlier edition in 1522, but Unger (p. 51) does not refer to any existing copy. In *R. A.* (p. 139) the title just quoted is given as that of the 1522 edition, but this seems to be an error.

7°. [Christoff Rudolff]. "Behend vnnd Hubsch Rechnung durch die kunstreichen regeln Algebre, so gemeincklich die Coss genen̄t werden. Darinnen alles so treülich an tag gegeben, das auch allein ausz vleissigem lesen on allen mündtlichē vnterricht mag begriffen werden. Hindangesetzt die meinüg aller dere, so biszher vil vngegründten regeln angehangen. Einem jeden liebhaber diser kunst lustig vnd ergetzlich. Zusamen bracht durch Christoffen Rüdolff von Iawer". The colophon is "Argentorati Vuolfius Cephaleus Ioanni Iung, studio & industria Christophori Rudolf Silesij, excudebat. Manus extrema operi data, mense Ianuario. Anno supra sesquimillesimum uicesimoquinto".

8°. [Christoff Rudolff]. "Kunstliche Rechnung mit der ziffer vnd mit den zal pfennigen, daraus, nit allain alles so sich in gemainen kaufmans hendeln zuetregt, sunder auch was zu silber vn̄ goldt rechnung, was zu schickhung des regels, was aynem muntzmaister: rechnung belangend: zugehorig, baide durch die Regl de tre (auch nicht on sundere vortail) vnd die Welhisch practick ausztzurichten, gelernnt wirt. Zu Wiēn in Osterreich allen liebhabern diser künst zu gemaynem nutz, durch Christoffen Ruedolf verfertigt. Getruckt zu Wiēn, im Iare nach der geburth Christi. 1526". The colophon is "Anno M.D. xxvi. Getruckt zu Wien̄ in Osterreich, durch Joannem Singriener".

9°. [Willibald Peer]. "Ain new guet Rechenbüchlein, Welches gar gewis vn̄ behend lernet nach der gemaynen regel Detre, mit sambt der welschen practigk, auff das kürtzist in die ziffer gesetzt, Mancherlay schöne vnd züwissen nottürfftige gerechnung, auff kauffmās handirung, es sey zum gewin

* Adam Riese published in 1518 "Rechnung auff der linihen gemacht durch Adam Riesen . . ." There was a second edition in 1525. See Unger, p. 49; *R.A.*, p. 139; Berlet, "Adam Riese, sein Leben, seine Rechenbücher . . ." 1892, pp. 1, 32. Unger knew of no existing copy of the 1518 edition. He gives the title of the 1525 edition, and states that it relates only to calculation by counters. Thus two different Arithmetics by Riese bear the date 1525; both were printed at Erfurt, and both were second editions.

oder verlust, vnd besonderlich überlandtrechnung. Gemacht zü Wienn in Osterreich durch Wilibaldum Peer, bürtig von Aystat". The colophon is "Getrückt zü Nürmberg durch Fridrich Peypus, im jar M.D. xxvij." Peer is not mentioned by Unger or Cantor: and the book is only incidentally referred to in *R.A.* (p. 156).

10°. [Petrus Apianus]. "Eyn Newe Vnnd wolgegründte vnderweysung aller Kauffmansz Rechnung in dreyen büchern ...durch Petrum Apianū von Leysznick, der Astronomei zü Ingolstat Ordinariū, verfertiget" (Ingoldstadt, 1527). A facsimile of the title-page is given on p. 156 of *R.A.* and the colophon on p. 155.

11°. [Johann Albert] "New Rechenbüchlein auff der federn, gantz leicht, aus rechtem grund, jnn Gantzen vnd Gebrochen, Neben angehefften, vnlangst, ausgelassnem Büchlein auff den Linien, dem einfeltigen gemeinen Man, vnd anhebenden der Arithmetica Liebhabern zu gut. Durch Iohan Albert, Rechenmeister zu Wittemberg, zusamen bracht, Auffs new mit allem vleis vbersehen, gemehret vnd gebessert. 1541". The colophon is "Gedruckt vnd volendet zu Wittemberg, durch Georgen Rhaw. 1542". This is the second edition. The first edition was published in 1534 (*R.A.*, p. 180). The edition of 1541 is the first mentioned by Unger (p. 55). The title of the 1561 edition is given in *R.A.* (p. 178).

12°. [Michael Stifel] "Arithmetica integra Authore Michaele Stifelio..." (Nuremberg, 1544). A facsimile of the title-page is given in *R.A.* (p. 225).

13°. [Michael Stifel] "Deutsche Arithmetica. Inhaltend. Die Haussrechnung. Deutsche Coss. Kirchrechnung. . . ." (Nuremberg, 1545). A facsimile of the title-page is given on p. 232 of *R.A.*

14°. [Gall Spenlin] "Arithmetica künstlicher Rechnung lustige Exēpel, Mancherley schöner Regeln Auff Liniē vnd Ziffern, vormals nie gesehen. Durch Gall Spenlin, Rechenmaister inn Vlm zü Trucken beschriben &c. 1.5.4.6. Gredruckt zü Augspurg, durch Hainrich Stayner". The colophon on p. CLV is "Getruckt zü Augspurg, durch Haynrich Stayner, im Iar M.D. XLVI.". To this book (in my own possession) the only reference I know of is in *R.A.*, where Eugene Smith reproduces the title-page of an edition of 1566 "Durch, Gallum Spänlin" (p. 274), printed at Nuremberg. He says (p. 271) that he knows of no other edition, but as the dedication is dated 1556 "this is possibly the date of the first edition".

In the edition of 1546 the dedication is dated 1545, so that there would appear to have been an intermediate edition between those of 1546 and 1566.

15°. [Adam Riese] "Rechenung nach der lenge, auff den Linihen vnd Feder...Durch Adam Riesen. im 1550. Jar" (Leipzig, 1550). A facsimile of the title-page is given on p. 251 of *R.A.*

16°. [Michael Stifel] "Die Coss Christoffs Rudolffs Mit schönen Exempeln der Coss Durch Michael Stifel Gebessert vnd sehr gemehrt... (Königsberg, 1553). A facsimile of the title-page is given on p. 259 of *R.A.* The colophon is dated 1554.

The use of the signs + and –, or equivalent words, in the sixteen books, §§ 43–63.

General statement, § 43.

§ 43. I have examined in these sixteen works* the occurrence of the signs + and –, of the words plus and minus, and of mer and weniger or minder. I have also looked for questions, similar to Widman's fig and pepper questions, in which the sign – or the word minus or weniger is used in connection with weights or with money.

Questions of the latter class (*i.e.* in which minus, minder, weniger, or –, occurs in the expression of the data) were found to be so numerous that it seemed better to consider them separately. I have therefore placed them together in §§ 64–75, after the general account of the different books and the use made in them of the signs + and – and the words plus, minus, &c. (§§ 44–63).

Before giving this general account it is convenient to recall that Widman defines + as mer and – as minus (§ 11), and that, except for this one use of minus, it does not occur again except in the rule of false. The word plus is used by him only in the rule of false.

Huswirt (1501), *Köbel* (1514), *Böschensteyn* (1514), § 44.

§ 44. The first two works, viz. 1° (Huswirt, 1501)† and 2°

* The numbers 1°, 2°, ..., 16° which are prefixed to the titles of the sixteen books in § 42 will be used to denote the books themselves. After the number denoting the book the name of the author and the date of the book will generally be added in parentheses.

† Since writing the account of the sixteen books I have met with an *Algorithmus* by Ambrosius Lacher de Merspurg, which was printed at Frankfort on the Oder between 1506 and 1510 (and is therefore earlier than any of the sixteen books, except 1°), which contains questions of the same kind as Widman's fig and pepper questions, the words plus and minus being used. This work will be described in § 76.

(Köbel, 1514) contain only the fundamental processes of arithmetic, and applications of the rule of three to examples which are not of mercantile significance. The third, 3° (Böschensteyn, 1514), is more comprehensive, and contains some comparatively complicated commercial examples, two of which are mentioned in § 65.

Grammateus (1518), §§ 45–46.

§ 45. 4° (Grammateus, 1518*) is a book of a higher class. Besides arithmetic it contains an algebra, the first published in Germany. In the arithmetic the signs + and – are used only in the rule of false (F vi′). The direction there given with respect to the result derived from a position is "if it is too large put +, if too small put –". (Ist zu viel setze + Ist aber zu wenig setz –). The signs + and – are freely used in the algebra, and are defined (G iii) as und and minder. (Vnd man braucht solche zaichen als + ist vnnd, – mynnder). In the algebra they connect the various cossic numbers in binomial expressions; and the rules for the addition, subtraction, and multiplication of quantities affected by the signs are given. Not only are the signs + and – used, but also plus and minus which occur in such expressions as

$$\frac{\text{2 ter: mi: 24 } N}{\text{16. 5t:}} \quad \text{and} \quad \frac{\text{72 pr: plus 72 } N\dagger}{\text{1 se: plus 2 pri:}}.$$

Grammateus gives a general explanation of the rule of false on F vi′ and F vii, but the numerous applications of it, in which the questions are solved both by the rule of false and by the Coss, occur later in the book (I v &c.) The signs + and – are always used, and the position and error are placed in the same line, as was done by Widman, but no cross is used.

§ 46. The word minus occurs quite early in the book (A vi′). It is there pointed out that the multiplication of two digits may be facilitated by the equalities

$$\overset{2}{\underset{2}{\text{mal}}} \text{ ist 20 minus 2 mal 8}$$

* Although I give the date 1518, I quote from the edition of 1521 (?), the title of which is given under 4° in § 42: and the page-references are to that edition.

† H iii and L iiii. Grammateus mentions the cossic names radix, census, cubus, &c., but the names which he actually uses are numerus, prima, secunda...abbreviated to N, pri, se, ter, quad...He seems to use plus and mi: only in fractions, which suggests that this is owing to the printer not having a type for + of the requisite size.

$$\begin{matrix}2\\ \text{mal ist 30 minus 3 mal 8, \&c.*}\\ 3\end{matrix}$$

Grammateus derived this rule from Peurbach, his words being "Vnd diese regel beschreybt vns Maister Georgius von burbach jn dem lateinischen algorithmo, gemacht fur die jungen studenten der hoen schuel zu Wien". The rule occurs in Peurbach's *Algorithmus*, under Multiplication, where, referring to it as 'regulam illam antiquam', he describès it as follows: "Quilibet digitus in aliquem digitorum multiplicatus in se producit eum numerum, qui manet postquam ab articulo à minori digito denominato, minor digitus tociens detrahatur quot sunt unitates a maiori digito ad denarium complendum, ut ter octo sunt triginta, demptis inde bis tribus. Postquam igitur digitorum omnium multiplicationes in promptu tenes, ad opus accedere potes".† I have thought it worth while to quote in full Peurbach's description of the rule in order to show the change in its mode of statement made by Grammateus, who after a brief description gives numerous examples of the rule, expressing them as formulæ and using the word minus (with a German context), just as – would now be used in $2 \times 2 = 20 - 2 \times 8$, &c. This must be one of the first instances in which the word minus is used in this manner in a printed book. In Peurbach's example the word demptis is used. Widman‡ also gave this rule, but he did not use a sign or the word minus or an equivalent.

The utility of the rule consists in replacing a multiplication of two digits, in which both are greater than 5, by a multiplication in which only one is greater than 5: and the rule is a very simple one, viz. to apply a cipher to the smaller digit and subtract from the number so formed the product of the smaller

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* Grammateus gives fifteen of these equalities, the last being

$$\begin{matrix}9\\ \text{mal ist 90 minus 1 mal 9.}\\ 9\end{matrix}$$

† I quote from the Leipzig edition of 1503, but the rule is substantially the same in the earlier Vienna editions.

‡ His description (p. 17′) is "Szo du eyn figur mit der anderñ ader mit yr selbst multiplicirñ bist sso secz albeg czu der kleynerñ ader sso sy gleich seyn zcu eyner welicher ess dan ist eyn 0 Vñd darnach wart wass zwischen der grosserñ ader irẽ gleichen vnd 10 ist. vnd szo manchmol 1 zwischen in peden ist. szo offt subtrahir die kleiner figur von der zal da fur du dã das 0 gesaczt hast. vnd wass dan do pleybet dz ist die zal darnach du gefragt hast Alsso hie yn diessem exempel 7 mol 8 Nu secz 0 fur die 7 alsso 70. vnd zwischen der grossern zal alss 8. vnd 10 ist 2. Darum subtrahir die kleyner zal alss 7 zwir vonn der. da fur du das 0 gesaczt hast. vnd ist 14 szo pleibt 56 vnd ist recht". The rule which Peurbach describes as antiqua occurs in Sacrobosco's *Algorismus*, where it appears in the form "quinam digitus multiplicat digitum, subtrahendus est minor digitus ab articulo suæ denominationis per differentiam majoris digiti ad denarium, denario simul computato" (Halliwell's *Rara Mathematica*, p. 12). In other manuscripts of Sacrobosco that have been printed the wording is slightly different.

digit and the complement to 10 of the larger, *i.e.* to multiply 7 by 8, 70 is written down, and the product of 7 and 2 subtracted from it. Of course by appending the 0 to the smaller number the product to be subtracted is made smaller.*

Grammateus (1521), § 47.

§ 47. 5° (Grammateus, 1521). This is a separate Arithmetic, derived from the arithmetical portion of 4°, but slightly differing from it. Neither the signs + or −, or the words plus and minus, occur. The rule of false is not given. A commercial question is quoted in § 66.

Riese (1525), § 48.

§ 48. 6° (Riese, 1525). Though technically a second edition, this is practically the *editio princeps* of this well-known work. The signs + and − occur only in the rule of false, and are very sparingly used there. They are defined (G viii) in the words "sagenn sie der warheit zuuil so bezeychenn sie mit dem zeychen + plus wu aber zu wenigk so bescreib sie mit dem zeychen ÷ minus genant";† but in the numerous examples that he gives of the use of the rule the words plus and minus are always used (and not + and −): for example, Riese writes the positions and errors as follows:

12	minus	3	
			5
24	plus	2	

At the end of the book, in a sort of appendix to the rule of false ('In Regula falsi', K iii), it is shown how a common factor may be divided out from both errors, or from both positions and the difference of the errors, &c.: and here the signs are used, but in a distorted form, the horizontal bar in both cases being extremely long.‡

* Widman and Grammateus also give the rule in which the two digits to be multiplied are placed the one under the other and their complements to 10 are placed to the right of them; then the product of the two complements gives the second figure of the product, and the first figure is obtained either as a cross difference or as the units' figure in the sum of the two given digits. Expressed algebraically the two rules are: if a, b be the two digits and a', b' their complements to 10, then $ab = 10a - ab'$, $ab = a'b' + 10(a - b')$ or $a'b' + 10(a + b - 10)$. Riese (1525) also gives both rules.

† This is, I think, the first instance in which ÷ is used instead of −. Riese uses − in his Arithmetic of 1550 (§ 61).

‡ I think we may infer from this that the words plus and minus were used in the examples on the rule of false instead of + and − owing to the want of suitable type for +.

There is a question in which weniger is used, and two questions of the same type as Widman's fig question in which centners and pounds are connected by the word minus (see § 67).

Riese's manuscript Algebra (1524), § 49.

§ 49. Riese wrote an algebra in 1524 which, however, he was not able to publish. It was discovered in the library at Marienberg in 1855, and an abstract of it was published by Berlet in 1860. This abstract was republished by him in 1892 in his pamphlet "Adam Riese, sein Leben, seine Rechenbücher und seine Art zu rechnen. Die Coss von Adam Riese. Von Realgymnasialrektor Professor Bruno Berlet. in Annaberg i. E." (Leipzig and Frankfort, 1892). The manuscript has the title "Adam Riesens seel. weiland Rechenmeisters zu S. Annaberg Anno 1524 auffgesetzte und mit eigener Hand geschriebene, aber niemals publicirte". In the abstract, as printed by Berlet, the signs + and − are freely used, and the word minus also occurs. In the dedication he refers to Widman and Grammateus.

Rudolff's Algebra (1525), § 50.

§ 50. 7° (Rudolff, 1525). This is a comprehensive work on algebra, showing a great advance on Grammateus. The signs + and − do not occur in the introductory chapters upon arithmetic, but first make their appearance in the first paragraph of chapter V (the first chapter of the Coss) where, referring to the addition and subtraction of numbers, he writes, "würt der zůsatz vermerckt bei dem zeichen +, bedeüt plus. der abzug bei dem zeichen − bedeüt minus (D ii)." He gives the rule for the addition, subtraction, and multiplication of quantities affected by + and − as in Grammateus. The signs + and − are freely used throughout the whole of the algebra in connection with numbers and the cossic signs, just as they would be at present. The rule of false is not given.

Rudolff's Arithmetic (1526), § 51.

§ 51. 8° (Rudolff, 1526). Rudolff's Algebra was published in 1525 and his Arithmetic in the following year. The signs + and − do not occur in the Arithmetic: nor do the words plus and minus occur in the text, but minus is frequently used in the examples. The rule of false is not given. The examples, which are of a mercantile character, are numerous, and include some of the same kind as Widman's fig question, and some

in which minus is used in connection with money (see § 68).* One of the latter deserves notice because it resembles Widman's question relating to the division of money in which the shares of the persons are '$\frac{2}{3}$ vnnd 6 mer', &c. (§ 27), but the mode of treatment is more satisfactory. The question is "Item drei haben zu tailen 138 flo. sol der erst haben $\frac{1}{2}$ vnd 6 flo. der ander $\frac{1}{3}$ vnd 4 flo. den dritt $\frac{1}{4}$ min⁹ 2 flo." (L 4′). In the solution 6 and 4 are taken from 138 and 2 is added, leaving 130, which is then divided in the proportions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.* Usage seems to have sanctioned the loose mode of statement which occurs in this question, for similar questions were given by Widman and Paciolo, and they continued to appear in books certainly up to the middle of the sixteenth century.† The meaning assigned to such questions by Paciolo and Rudolff seems to be the only admissible one. As in some other questions of the period the solution consisted as much in finding a suitable interpretation of the question as in the arithmetical work. In Widman's interpretation minus simply denotes a subtraction; in Paciolo's the minus is at first treated as a direction to pay, 'vnd' denoting something additional to be received.

§ 52. De Morgan was acquainted with this work of Rudolff's, but he was unaware of the existence of any copy of the Algebra which Rudolff had published in the preceding year.* A second edition of this Algebra was published by

* Rudolff uses the words dragma, radix, census, cubus, and the cossic symbols, which remained in use for so long afterwards (not the N, pri, se, ter, ..., of Grammateus).

* When a sum of money was to be divided among three partners, so that their shares were to be $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, the accepted meaning was that the division was to be made in these proportions, but when the money was to be divided so that the shares were $\frac{1}{2}+6$, $\frac{1}{3}+4$, $\frac{1}{4}+2$, this might mean either that a number λ was to be taken arbitrarily and the money was to be divided in the proportions $\frac{1}{2}\lambda+6$, $\frac{1}{3}\lambda+4$, $\frac{1}{4}\lambda+2$, or it might mean that after the partners had received respectively 6, 4, 2, the remainder was to be divided among them in the proportions of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$. The latter is the more satisfactory interpretation of an ambiguously expressed question, as it leads to a definite solution. Widman, however, adopted the former interpretation, and would have taken λ to be some number containing 2, 3, 4, as *e.g.*, 24.

† Besides the questions referred to in the note to § 27 (p. 19), Paciolo gives two others (p. 150′) in which 100 is to be divided among two persons so that one has 'la. $\frac{1}{2}$. $\overline{\text{m}}$. 4' and the other 'el. $\frac{1}{3}$. $\overline{\text{m}}$. 2' and in which 10 is to be divided so that one has 'la. $\frac{1}{2}$. $\overline{\text{m}}$. 3' and the other 'el. $\frac{1}{3}$. $\overline{\text{p}}$. 4". In the solutions of these questions 106 and 9 are divided in the proportion of 3 to 2 in order to obtain '$\frac{1}{2}$' and '$\frac{1}{3}$'. Feliciano (*Libro di Arithmetica*, 1526) has a similar question (H 3′) in which 100 is to be divided and the shares are 'la mita piu .3.', 'el .$\frac{1}{3}$. piu .5.', and 'el $\frac{1}{4}$ piu .2.'. A more elaborate question of the same class in given in 15° Riese (1550). See § 62.

* "The work in which + and − were applied to algebra is taken to be the first edition of Christopher Rudolff's *Die Coss*, 1524. This first edition is lost: no copy of it is known to exist. A Latin translation, made in 1540, is said to exist in the Imperial Library at Paris . . ." After mentioning a reference made by Rudolff in his Arithmetic to his Algebra of the preceding year, and referring to Stifel's edition (of 1553) he concludes "I shall not feel satisfied about the first edition of Rudolff until either a copy is produced, or a full account of the Latin manuscript is published" (*Camb. Phil. Trans.*, vol. xi, p. 209, 210).

Stifel (see § 63), but De Morgan did not consider that the free use of + and − in that edition justified the inference that the signs had occurred in the same manner in the first edition, as they might have been introduced by Stifel. He writes, "In the *Künstliche Rechnung* of 1526 Rudolff does not show a single mention or use of + and −: his signs are × as a pair of guides in the addition of fractions, and × = in the rule of three. If he had had his head as full of + and − as a man must have had who had published in 1525 a *Die Coss* like that of 1554, this would have been very strange."* This, however, is what actually happened; for in the *Rechnung durch die kunstreichen regeln Algebre* of 1525 the signs + and − are freely used, and yet they do not occur at all in the *Künstliche Rechnung* of 1526: nor do they occur in the arithmetical portions of the former work, Rudolff confining them to algebra.

Peer (1526), § 53.

§ 53. 9° (Peer, 1526). The remarks upon the contents of 8° (Rudolff) apply also to this arithmetic, except that weniger and minder are used instead of minus in the examples (see §§ 69–70).

Apianus (1527), §§ 54–55.

§ 54. 10° (Apianus, 1527). This is a good commercial arithmetic, written by Petrus Apianus the astronomer.† It shows individuality, and differs in essential respects from Widman, Riese, or Rudolff. Quite early in the book (E vi′) there is a question, formed entirely on the model of Widman's fig question, in which the signs + and − are introduced and defined as in Widman. In this question, 9 barrels of oil, whose weights are expressed in centners and lbs, connected by the signs + and −, are defined in the words "Das + bedeut mehr, das − bedeut minder". The tare on each barrel is given, and it is explained that the total tare is to be subtracted from the sum of the weights. The form of the question is exactly similar to that of Widman's, but Apianus first finds the sum of the weights (taking into account the positive and negative numbers of lbs) and then subtracts the tare from this total, whereas Widman added the sum of the lbs having a negative sign to the tare, and subtracted the total from the sum of the centners and the lbs with a positive sign.

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* *Id.*, p. 210.

† I do not know why De Morgan refers to the author of this arithmetic as a 'writer printed by Apian' (*Id.*, p. 208).

Although + and − are thus introduced so early in the book this single question stands alone, and the signs do not occur in questions of a similar kind (such as those in Rudolff and Peer), the word minder, and in one case minus being used instead. The question involving + and − is quoted in full in § 71, as also are some of the other questions.

§ 55. After the question on E vi′ the signs next occur on M i′, when they are defined anew in the rule of false. The definition there given is "zü zeitten meher oder minder, was zü vil ist vermerck mit dem zeichen plus +. Ist zü wenig, mit dem zeichenn minus −", and he gives the rule for addition and subtraction in words and in the form

$$\underset{\text{Subtrahir}}{\overset{+}{+}\ \ S\ \ \overset{-}{-}} \qquad\qquad \underset{\text{Addir}}{\overset{+}{-}\ \ A\ \ \overset{-}{+}}$$

In the examples on the rule of false the signs + and − are used, and Apianus places the error below the position (as was done by Borgi and Paciolo, though they of course did not use the signs).* When several results are required in the question he deals with them all simultaneously in the process (instead of determining only one, and deducing the others), so that his diagrams contain a number of guide lines radiating from the errors instead of the simple cross.

Albert (1541), § 56.

§ 56. 11° (Albert, 1541) is a commercial arithmetic with a great many examples. The sign − first occurs in a question on D viii′, where a barrel of butter contains '2 centner − 4 pfund', that is, 216 lbs, as here a centner is 110 lbs. It next appears on I iii, where a weight is given as '12 cent. 4 stein − 6 pfund',† and there are a number of similar questions in which − occurs. It is also used in connection with money, as *e.g.* '9 fl − $3\frac{1}{2}$ ort' (L iii′); '17 gr $\frac{1}{2}$ − 2ϑ' (M iiii) (see § 73). The sign − is used always, and neither the word minus nor any German equivalent occurs.

The signs + and − are defined in the rule of false (M vi) as follows: "Vnd wo solche zwo Falsche zaln, etwas mehr (denn die Frag begert) bringen, So mach von stund, nach der Falschen zal, diese linie mit durchgezognem stricklein, also —/— Vnd bedeutet Mehr . . . Wo aber zu wenig, zeuch nach

* This is the first German printed book in which I have seen the error so placed.

† A stein is 22 lbs ($\frac{1}{5}$ of a centner).

der Falschen zal diese linie — Vnd bedeut weniger". In the diagrams in the rule of false the position and error are written in the same line, thus

24—/—10 (22

16——12

It is evident that there was no type for +, and that the symbols used were put together at the printing office by combining — and /.

Stifel's Arithmetica Integra (1544), § 57.

§ 57. 12° (Stifel's *Arithmetica Integra*, 1544) is a treatise on arithmetic and algebra showing great power and originality. The arithmetic is not commercial, and there are no questions of a mercantile character.

The signs + and – appear, without explanation, on p. 38, where it is mentioned that all numbers can be expressed by powers of 3: thus, 1, 3 – 1, 3, 3 + 1, 9 – 3 – 1, 9 – 3, 9 + 1 – 3, &c. They do not occur again in the arithmetic: even in the rule of false the words plus and minus are used.

In the algebra (p. 110), under the heading "De signis illis duobus + & –", he writes "Quando addenda sunt duo incommensurabilia, uel duo aliqua, quorum proportio ignoratur (ut in cossicis numeris ferè ubique fit) tunc interponimus signum hoc + ipsis addendis, dicimusque ita completam esse additionem. Et hac ratione uocatur signum additorum.

Simili ratione uocamus signum hoc –, signum subtractorum: dum em̄ subtrahere uolumus aliquid, ab alio sibi incommensurabili, aut ab eo, cuius proportio ad subtrahendum est ignota, tunc utimur illo signo, ut in exemplis tanquam compositorum uidebis paulo inferius".

He repeats this explanation again on p. 248′, but with an addition, viz. "Sciendum tamen, quod duplici necessitate utimur signis illis. Primo enim utimur illis in numeris talibus, quorum proportio præcise dari non potest: ut in numeris irrationalibus. Secundo utimur eis in numeris talibus, quorum proportio ignota est, & si præcisa sit: ut in numeris cossicis, dum numeros quærimus nobis absconditos. Sed præter necessitatē hanc duplicem, utimur eis commoditatis gratia, ut aliquid per ea monstremus aut doceamus, quēadmodum me hoc loco uti videbis, non necessitatis sed commoditatis causa".

Thus Stifel considers that the use of the signs is to connect by addition and subtraction quantities for which the actual addition and subtraction cannot be performed: but then he

adds that they can be used whenever it is convenient. Stifel's use of the signs is exactly the same as at the present day.

In the dedication of his third book (which relates to the rules of algebra*) Stifel mentions his obligations to Rudolff and Riese, whom he has never seen but for whose works he has a great admiration.

Stifel's Deutsche Arithmetica (1543), §§ 58–59.

§ 58. 13° (Stifel's *Deutsche Arithmetica*, 1545) consists of an arithmetic and algebra. The former consists of only 16 leaves, and there are no commercial questions. In the algebra the signs + and − are used, but not the cossic symbols; in fact, the work is an attempt to teach algebra (the solution of equations) without the use of the cossic symbols, the word 'sum:' replacing the cossic symbol for res or cosa, 'sum: sum:' replacing census, &c., with other developments.

§ 59. Libri was of opinion that in this work Stifel claimed for himself the invention of the signs + and −, the remarks in his sale catalogue with respect to the book† being: "Stifel uses the sign + for addition and − for subtraction, and distinctly states that whenever you see + you may read *Und oder Mer* (*Et* vel *plus*) and where you see − *Weniger oder Minder* (*paucius* vel *minus*). He also distinctly claims these signs as his own invention, 'Darumb so gedenck nur nicht, das dise ding schwer seyen zu lernen, oder zubehalten, vnd ist doch die gantz sach, DIESER MEINER ZEICHEN hiemit gantz auszgericht vnnd an tag gebracht'".

We know that this could not have been Stifel's meaning, as he was familiar with Rudolff's algebra, nor need his words convey this impression. The following are the sentences in in which + and − are explained:

"So ich aber ein gerechnete zal‡ (als 3 oder 3 fl. &c.) soll addiren zu einer vngerechnetē zal (als zu 2 sum: oder 3 sum:) so muss ichs thun durch das zeichen +, welchs ich setzē musz zwischen sie, als 2 zu 3 sum: machen 3 sum: + 2. das machstu denn also lesen, 3 summen vnd 2. Denn wo du dises zeichen + findest, da magstu an stat des selbigē zeichens lesen disz wörtlein Vnd oder Mer.

"Also auch, so du solt ein gerechnete zal Subtrahiren von eyner vngerechneten, als 7 von 10 sum: so steht das exem-

* p. 226′.

† No. 594 (p. 73) of the catalogue referred to in the note on p. 1.

‡ A 'gerechnete zal' is equivalent to a known quantity, and an 'ungerechnete zal' to an unknown quantity.

plum also, 10 sum : – 7. Das magstu also lesen, zehen Summen, weniger 7. Denn wo du dises zeichen – findest, magstu darfur lesen, Weniger oder Minder, deñ es ist ein zeichen des subtrahirens, gleich wie + ist ein zeichen des addirens. Item so ich soll 2 sum: subtrahiren von 70. so steht das exempel also, 70 – 2 Sum :" (p. 21′).

Stifel then in several other paragraphs describes another notation, explaining again the use of + and –, and finally concludes (p. 22) the whole account with the sentence "Darumb so gedenckt . . . gebracht" quoted by Libri.*

It seems to me that the words 'these my signs' mean no more than 'these signs which I am using'. As a fact, the notation sum :, sum : sum :, sum : *A*, &c., was invented by Stifel, but I do not think that his words were intended to assert a claim to it.†

Spenlin (1546), § 60.

§ 60. 14° (Spenlin, 1546) is an ordinary commercial arithmetic. The signs + and – are used only in the rule of false and are defined (p. cxlii) "bedeut das zaichen + züuil, vnd das ÷ zü wenig". In the diagrams the position and error are written in the same line and a large cross is used: so that the signs + and ÷ stand between the arms of it (see § 86).

The words plus and minus are not used. The word minder occurs in the data of three questions, and is the word used in connection with both weights and money. These will be referred to in § 74.

It will be noticed that Spenlin follows Riese in using ÷ for minus.

Riese's Rechenung nach der lenge (1550), § 61.

§ 61. 15° (Riese's *Rechenung nach der lenge*, 1550)‡ is a comprehensive arithmetic not unlike Rudolff's *Künstliche*

* The words 'dieser meine zeichen', which Libri printed in capitals, are not emphasised in the original.

† In 1857 Cantor, referring to the fact that the signs + and – were still generally attributed to Stifel (although Drobisch had shown that they were used in 1489), quoted the *Deutsche Arithmetica* as affording evidence to show that they had an older origin ('Indirect gesteht Stifel selbst, dass die Zeichen + – älteren Ursprunges sind'). In the passage in question Stifel says that just as one adds by the sign + ('wie man addiret durch das Zeichen +') so he multiplied by the sign M and divided by the sign D (*Zeitschrift für. Math. und Phys*, vol. ii. p. 366).

‡ Berlet states on p. v of his life of Adam Riese, prefixed to the tract referred to in § 49, that this work was completed in 1525 ("Ausserdem hatte Riese 1525 zu Annaberg eine 'Arithmetic' vollendet, welche aber erst 1550 in Leipzig in 4to unter dem Titel edieret wurde: Rechenung nach des lenge . . .") Berlet does not give any authority for the statement, and I think it safer to assign to this Arithmetic the date of its publication. The first part (p. 1) is headed "Rechnung auff den Linien nach der lenge durch Adam Riesen im M.D.L.", and the second part (p. 47′) "Rechnung nach der lenge mit der Feder. Durch Adam Riesen im 1550".

Rechnung (§ 51) with the *Exempel Büchlin* (§ 72) added, but more complete. Many difficult questions are carefully worked out, and the processes fully explained. As an arithmetic it is excellent and much superior to any of its predecessors. It shows that Riese's enduring reputation as a 'Rechenmeister' was well deserved.*

The signs + and − are used in the rule of false and are explained as plus and minus on p. 167 ("mit dem zeichen + das ist plus . . . mit dem zeichen − das ist minus"). The sign − had been previously used, but not the sign +. Both + and − are used in the concluding portion of the book, which relates to gauging. The sign − first occurs on p. 114 in '4 − 1' mentioned in the next section (p. 46). The word minus occurs frequently all through the book, and occasionally the word weniger. The word plus occurs twice before the rule of false.

The first occurrence of either minus or weniger is in the rule for the sum of a geometrical progression (p. 13′) where the denominator is described as the 'ubertretung weniger 1' (ubertretung being the common ratio). It next occurs in question 108, which begins "Einer kaufft drei schock hüner weniger 13 . . ." This question occurs twice,† and in both solutions the word minus is used. In the first solution (p. 22′) Riese gives as a direction "Nim ab das minus bleiben 167" (which is the first use of minus in the book), and in the second solution (p. 82′) he places 13 under 180 in order to subtract it, and writes minus by its side. In question 118 (p. 23′) he gives the weight of a sack of pepper as '3 ct minus 17 lb', and minus is used in many other questions afterwards.

On p. 127′, when he has to multiply 7 fl $3\frac{1}{2}$ ort by 29 he takes the money to be '8 fl − $\frac{1}{8}$', multiplies the 8 florins by 29, and subtracts $\frac{1}{8}$ of 29 florins.‡ On p. 137′ he multiplies in a similar manner '3 minus $\frac{1}{8}$ fl' by $19\frac{1}{3}$ and on p. 138 '2 − $\frac{1}{8}$ fl' by $210\frac{3}{4}$. On p. 139 in an example where the

* Berlet (*l.c.* p. vii) says that it was generally held that anyone who had completely worked through Riese's *Practica* should be regarded as a master of calculation. The *Practica* extends from p. 106 to p. 182, and contains so many difficult questions of various kinds that this opinion was fully justified.

† A great many of the examples occur twice, as pp. 1–47 relate to calculation 'auff den Linien' (by counters) and the subsequent portion to calculation 'mit den Feder': and the examples in question are solved by both methods. This also applies to the rule for the summation of a geometrical progression which occurs on p. 53 as well as on p. 13′. I shall refer to questions by their numbers, adding the page or pages when it seems desirable. As questions which occur twice are not always identical either in wording or spelling or in the directions for their solution, the reference sometimes applies to only one version of the question.

‡ He also obtains the result by multiplying 7 fl $3\frac{1}{2}$ ort by 4 and 7 and adding the original quantity, but by an error this is indicated (at the side) by '4', '7 − 1' instead of '4', '7 vnd 1'.

weights are '3 ct minus 11 lb vnd $2\frac{1}{2}$ ct 5 lbs' and the 'tara' is 19 lb, Riese writes down '300 − 11' and places 255 under 300 and 19 under 11; he thus obtains by addition 555 and 30 and, by subtraction, 525 as his multiplier.

On p. 124 he finds a fourth proportional to 8, 3 fl 5 gr 2℈, and 6, by subtracting from the middle term $\frac{1}{4}$ of itself, writing 8 minus 2 at the side to indicate that he treats 6 as 8 − 2.

Similarly on the next page (p. 124′), in finding a fourth proportional to 6, 5 and 17, he subtracts from 17 its sixth part and writes '6 − 1' as indicating the process: and on p. 125, in finding a fourth proportional to 5 lb, 9 fl 7 gr, and 9 lb, he doubles 9 fl 7 gr and subtracts one fifth of it, indicating the process by '10 − 1'. In another example, on p. 126, he treats 29 as 30 − 1, which he writes at the side. On p. 128, where he has to multiply 17 gr by 37, he multiplies by 42 and subtracts 5 times 17, writing '42 − 5' at the side. His object here in multiplying by 42 is for convenience in reducing to florins, as there are 21 groschen in a florin: and on pp. 128′–130 he treats 19 gr, 18 gr, 39 gr, 14 gr, 40 gr as '1 fl − 2 gr', '1 fl − 3 gr', 2 fl − 3 gr', '1 fl − 7 gr', '2 fl minus 2 gr.' On p. 131′, where he has to multiply 17 gr by $39\frac{1}{2}$, he multiplies '2 fl minus $2\frac{1}{2}$ gr' by 17, and on p. 132 he replaces $41\frac{3}{4}$ by '2 fl minus $\frac{1}{4}$ gr'.*

§ 62. The word plus first occurs on p. 159 in the solution of the question "Fünff haben zu teilen $124\frac{1}{2}$ fl dem ersten gebürn $\frac{2}{3}$ − 12 fl, dem andern $\frac{1}{4}$ vnd 10 fl, dem dritten $\frac{5}{6}$ minus 24 fl, dem vierden $\frac{3}{8}$ vnd 6 fl, vnd dem fünfften $\frac{2}{5}$ minus 7 fl wieuil gebürt jedem". Riese adds the minus numbers which amount to 43 and subtracts 16 leaving 27 which added to $124\frac{1}{2}$ gives $151\frac{1}{2}$ fl as the sum to be divided in the proportions $\frac{2}{3}, \frac{1}{4}, \frac{5}{6}, \frac{3}{8}, \frac{2}{5}$. His words are "Summir das minus wirt 43 desgleichen das da mehr ist als 16 nim von einander bleiben 27 die gib zu dem gelt, das sie zu teiln haben als $124\frac{1}{2}$ wirt $151\frac{1}{2}$". Dividing $151\frac{1}{2}$ in the proportions 80, 30, 100, 45, 48 he obtains 40, 15, 50, $22\frac{1}{2}$, 24, from which he derives the results '40 − 12', '15 vnd 10', '50 − 24', '$22\frac{1}{2}$ vnd 6', '24 − 7' (the direction being 'das minus nim herab, das plus gib hinzu wie jedem vorzeichent'), that is, 28 fl, 25 fl, 26 fl, $28\frac{1}{2}$ fl, 17 fl.

Thus, in the data, quantities to be added and subtracted are indicated by 'vnd' and 'minus' or '−'; in the solution the sum of the former is designated 'das da mehr ist' and 'das plus', and the sum of the latter is twice designated 'das minus'.

The second occurrence of plus is on p. 162 in the direction "die andern mach durch plus vnd minus".

* Misprinted '2 fl minus $\frac{1}{4}$ eln'.

A peculiarity of the work, which I have not found in any previous Arithmetic, is that the sign — or the word minus is used in the actual working out of examples. Thus on p. 117 he shows how in order to divide by 47, we may divide by 48 (*i.e.* by the factors 6 and 8) and deduce the required result, and he indicates the process by writing '6' and '8 minus 1' at the side.*

On p. 114, in order to multiply by 23, he multiplies by 6 and 4 and subtracts the original quantity: this he describes as 'Setz 6 mal 4 minus 1', and he indicates the operations by writing '6' and '4 − 1' by the side of the two multiplications. On the next page, to multiply by 43 he multiplies by 5 and 9 and subtracts twice the original quantity, writing at the side '5' and '9 minus 2'. On p. 132′, to multiply by $39\frac{5}{8}$, he multiplies by 5 and 8 and then subtracts $\frac{3}{8}$ of the original quantity, denoting the operations by '5' and '8 minus $1\frac{1}{2}$',† and on p. 133′ in multiplying by $23\frac{1}{2}$ he indicates the operations by '4' and '6 minus $\frac{1}{2}$'.

Stifel's edition of Rudolff's Algebra (1553), § 63.

§ 63. In 16° (Stifel's edition of Rudolff's Algebra, 1553) the signs + and − are used as in the original edition (1525), *i.e.* very freely in the whole of the algebra and not at all in the arithmetic which precedes it.

In his appendix to Chapter V., in which Rudolff explains the cossic signs and the signs + and −, Stifel remarks that Rudolff's algorithm of the cossic signs includes three algorithms, viz. of ordinary numbers, of the cossic numbers, and of + and −. Of the third algorithm he says that it is particularly powerful and happy (lustige) not only in the cossic numbers and the surd numbers but in all kinds of numbers, as, for instance, in ordinary numbers or fractions ("So kompt nu der Algorithmus der zweyen zeychen + vnd − dareyn. Das ist ein sonderliche gwaltige vnd lustige sach, nicht allein bey den Cossichen zalen, wie der vorgehnde Algorithmus meldet. Auch nicht alleyn bey den surdischen zalen, wie der nachfolgende Algorithmus dess zehenden Capitels wirt melden, sondern bey allerley zalen. Als bey zalen vnd Brüchen

* The general process which Riese here applies in an example may be explained as follows: if it is required to divide a number n by a divisor p, and if $p+a$ (having factors) is a more convenient divisor, then let q be the quotient and r the remainder, when n is divided by $p+a$: let $aq+r$ be formed and divided by p, giving q' as quotient and r' as remainder: then the quotient when n is divided by p is $q+q'$ and the remainder r'. The process depends upon the equations

$$x=(p+a)\,q+r=pq+aq+r=pq+pq'+r'=p\,(q+q')+r'.$$

† *i.e.* 8 minus $1\frac{1}{2}$ virtheil, a virtheil being $\frac{1}{4}$.

gemeyner benennung etc.", p. 71). Stifel then gives the rule of signs for addition, subtraction, multiplication, and division.*

Questions in the sixteen books in which − or its equivalent occurs in the data, §§ 64–75.

Introductory remarks, § 64.

§ 64. It is evident from the account which has just been given of the occurrence of the signs + and − in these sixteen books that they at once became a part of algebraical notation, but that they were very sparingly applied to arithmetic except in the rule of false (which is more closely allied to algebra than to arithmetic and in which the signs do not denote addition and subtraction); but before considering further the uses of these signs it is desirable to examine the numerous questions of the same type as Widman's fig and pepper questions (§§ 11 and 16) which occur in these books: and to this I now proceed.

As mentioned in § 38 this investigation is of interest, as it should throw light on the origin of these questions, *i.e.* whether they were derived from actual mercantile transactions, or were merely invented by arithmeticians as exercises in calculation.

Böschensteyn (1514), § 65.

§ 65. Böschensteyn (1514) has no question in which minus occurs, but he has commercial questions involving more than the simple rule of three. One of these relates to tara, though the name is not used, and the other is given under *Regula fusti.* In the first (C iiii') a person buys 3 barrels of oil weighing 345 lbs, 362 lbs, 351 lbs, and he takes off for the wood 11lbs on the centner (schlecht ab für das holtz 11lb am ct) and sells one centner pure (lauter) for 6 fl $1\frac{1}{2}$ ort. The solution shows that the meaning is that he sells 111 lbs for $6\frac{3}{8}$ fl. In the question on the *Regula fusti* (C vi') he buys 27 ct 81 lbs of cloves and pays 11 ss 3 hlr for 1 lb of the pure cloves, and 21 heller for 1 lb of the fusti (stalks), and 100 lbs contain (halten) 13 lbs of fusti.*

* For the addition and subtraction of numbers with unlike signs his rule is that in addition the smaller number is to be taken from the larger and in subtraction the numbers are to be added, and that the sign to be affixed is given by the syllable Go, viz that in addition the sign is that of the greater (Grosser) number, and in subtraction it is that of the upper (Ober) number (pp. 73, 73').

* This question is taken from Widman, p. 93' (see § 19), 2 ss − 3 hlr being replaced by 21 hlr. The same question also occurs in Peer, F 4', where also the price of the fusti is given as 21 hlr (see § 69).

Grammateus (1518 *and* 1521), § 66.

§ 66. In 4° and 5° (Grammateus, 1518 and 1521) under the heading *Regula fusti* there is the question "Ich hab kaufft 6 cetner pfeffer ye 1. cētner lauter vmb. 50. fl. R hielt. 1. centner tara. 5. lb. ist die frag wie tewr kombt der pfeffer".* In the solution the 30 lbs tara is subtracted from the 6 centners, leaving 5 cent. 70 lbs. This question occurs on D vi′ of 4° (the arithmetic and algebra) and on C i′ of 5° (the arithmetic). This is the earliest use of tara I have met with in a German book (see § 92). There is no other commercial question of any importance; the word minus is used in 4° (§§ 45, 46), but not in 5°.

Riese (1525), § 67.

§ 67. In 6° (Riese, 1525) there are three questions involving weniger or minus. The first (C viii) is "Item eyner kauff ein schogk hüner weniger 9 halb zu 14 vnd halb zu 15 pfen: facit 2 floren 19 grosz: 7 pfen: vnd ein heller". Here a schock is 60, so that the number of fowls is 51.

The next question (D vi′), which is of the same type as Widman's fig question, is as follows: "Item vier lagel mit Seyfen wegen 3 cen: minus 13 pfundt, 4 cent: ein pfundt, 4 cen: minus 28 pfundt vnd 3 cen: minus 11 pfundt tara vff ein cent: 10 pfundt vnnd kost ein pfundt lauter 16 pfen: ein halben facit 80 flo: 6 gro: 3 pfen: den flo: fur 21 grosz: vnd ein gros: fur 12 pfen".†

There is also a similar example on E vii′, in which "3 Vhesser mit schmer wegē 4 cen. minus 13 pfundt 3 cent. 28 pfundt vnd 5 cen. 11 pfu".

Thus in Riese we have the first examples (in any of the books mentioned) in which centners and lbs are connected by the word minus, but minus is not used in connection with money. See, however, § 76.

Another question (E v′) may be mentioned because of the use of mer for vnd. "Itē eyner kaufft 43 pfundt Saffran das pfu: fur 3 flo: 10 ss mer‡ 58 pfundt Negelein ein pfu: fur 16 ss vn̄ 75 pfundt Ingwer . . ."

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* D vi′ in 4° (from which I quote the question) and C i′ in 5°.

† The word tara is first used on D iiii′ in the question "Item ein stumpff Saffran wigt 38 pfundt 16 lot tara 9 lot vnd man gibt $3\frac{2}{3}$ pfunt fur $8\frac{3}{4}$ floren facit 91 floren 4 schil. 0 heller $\frac{84}{88}$ teyl". In the solution it is explained that the tara is to be subtracted from 38 lbs 16 lot, leaving 38 lbs 7 lot.

‡ This use of mer occurs frequently in Rudolff (1526) and subsequent books.

Rudolff (1526), § 68.

§ 68. In 8° (Rudolff, 1526) the first use of minus is in the question (I 3′) "Item wañ $\frac{1}{2}$ von $\frac{3}{4}$ aus $\frac{5}{6}$ ainer eln, costet 2 flo. min⁹ $\frac{1}{3}$ von $\frac{1}{4}$ ains florens, was weren werdt 25 eln $\frac{1}{2}.\frac{1}{3}.\frac{1}{5}$. facit 159flo. 5ss. 11$\partial\frac{1}{15}$". The question is: if $\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}$ of an ell costs $(2-\frac{1}{3}\times\frac{1}{4})$ fl what do $(25+\frac{1}{2}+\frac{1}{3}+\frac{1}{5})$ ells costs, *i.e.* if $\frac{5}{16}$ of an ell cost $1\frac{11}{12}$ florins, how much do $26\frac{1}{30}$ ells cost?

It is clear that this question could not have arisen in the way of trade, and it must have been constructed merely as an exercise in fractions and the rule of three.

The next question, on the next page, I 4, is of the same class. "Item wann $\frac{2}{3}$ von 10. weren 15 min⁹ $\frac{1}{2}$ von $\frac{1}{3}$ aus 26. was weren $\frac{3}{4}$ von 18. Facit $21\frac{3}{5}$", *i.e.* $\frac{20}{3}:\frac{32}{3}::\frac{27}{2}:\frac{108}{5}$.

When minus first occurs between concrete quantities it is in connection with money, viz. (I 4′) in the question "Item 1 stumpf saffran wigt 37 lb $\frac{1}{2}$. thara 9 lot fur den stumpf, costen je 3 lb $\frac{1}{4}$. 20 flo. min⁹ 1 ort. facit 226 flo. 3 ss. 6 h. $\frac{3}{26}$". This is the first question in which thara occurs, and it is explained that it is to be subtracted, and what is left is saffron. [There are 32 lot in a lb, so that the question amounts to finding the cost of 37 lb 7 lot if $3\frac{1}{4}$ lb cost $19\frac{3}{4}$ fl.]

In two other questions on I 4′, which relate to skins, the price per 1000 is given as '60 flo. min⁹ $\frac{1}{2}$ ort'. After a question on I 5, in which weights are given in centners and lbs, Rudolff gives one in which minus occurs, viz. "Item 3 trucken mit saiffen, wegen 3 ceñ. mi⁹ 12 lb. 4 ceñ. 14 lb, 5 ceñ. min⁹ 14 lb. thara fur holtz 10 lb auf 1 ceñ..." There is a similar question relating to honey on I 5′, and on I 6 there is a question in which a florin is to be taken to be 8 ss minus 6∂ (dē flo. gerechnet fur 8ss mi⁹ 6∂.), *i.e.* the florin is to be 7 shillings and 24 pence. In four other questions sums of money are given as '2 flo. min⁹ 1 ort $\frac{1}{2}$' (I 8′), '4 ducaten min⁹ 7 ss', '14 horn gulden min⁹ 2 ss' (K 4′), '50 flo. min⁹ 1 ort' (L 2).

Peer (1527), §§ 69–70.

§ 69. 9° (Peer, 1527) has questions of the same kind as Rudolff, but he uses weniger and minder instead of minus.

In the first question on D 1 a person buys 3 sacks of pepper "der erst wigt 1 centner 43 lb. der ander wigt 1 cent. 74 lb. der drit wigt 2 ce. weniger 13 lb"*, &c.

In an example on D 3′ three barrels of honey weigh "4 c. 40 lb. 5 c. minder 22 lb. 3 c. minder 8 lb. thara auff 1 c 10 lb."

* Further on in the book (I 5) this question is worked out in detail as a specimen of 'Die Welsch practigkh'.

On D 7 there is a much more elaborate question of the same kind. "Item einer kaufft 12 lagel saiffen, die erst die wigt 304lb $\frac{1}{2}$, die ander wigt 400lb. weniger 15lb $\frac{3}{4}$ drit 312lb $\frac{1}{4}$. viert 298lb $\frac{3}{4}$. fünft 500lb weniger 27lb $\frac{1}{2}$. sechst 300lb. minder 3lb. Siebet 430 lb $\frac{1}{4}$. achtist 398 lb $\frac{1}{2}$. Neunt 400 lb. weniger 35 lb. zehent 390 lb. aylfft 289 lb. zwölfft 400 lb. weniger 17 lb. $\frac{1}{2}$. Thara für 1 lagel 20. $\frac{1}{2}$ lb kost 1 cen. 4 fl. 7 ss. frag was die 12 lagel kostē. F. 198 fl 6 ss 18 ϑ $\frac{9}{20}$".

§ 70. There is another question of the same kind on D 7′, in which 7 barrels of soap weigh '340 lb $\frac{1}{4}$', '500 lb weniger 13lb. $\frac{1}{2}$', '435lb. $\frac{1}{2}$', '400lb weniger 11lb $\frac{3}{4}$.', '210lb $\frac{1}{4}$.', '300lb.', '403 lb $\frac{1}{2}$.'; and on D 1′ there is a question in which minder is applied to money, viz. "Item 1 sack pfeffer wigt 2 cent. $\frac{1}{2}$ weniger 9 lb. vn̄ kost 1 lb 8 ss mīder 3 h. vn̄ geet ab 3 lb. $\frac{3}{4}$. F. 91 fl 18 ss 8 h $\frac{1}{4}$". Here the sack of pepper weighs 241 lb and 1 lb costs 7 ss 9 h and the tara for the sack is $3\frac{3}{4}$ lb.

On D 7 minder is again applied to money in a question in which the price of 1lb is '2fl minder ij ort'; and this occurs again in a question on G 4′ in which mer is used for and: viz. a person buys "33. zobel, je 1 zimer vm̄ 27 fl. mer 265 harn pelg, ein hundert vmb 7fl. minder j ort, mer 735 lasset..."

It will be noticed that in the questions on D 7 (quoted in § 69) and D 1′ (quoted in the present section) both weniger and minder are used in the same question. On D 5 Peer has a question about a shock of fowls similar to Riese's. "Itē 1 schock hünner weniger 7 hon, halb zu 12 vnd halb zu 13 ϑ".

On D 4 and D 4′ he takes Rudolff's questions about skins, merely changing the price per 1000 from '60 flo. min⁹ $\frac{1}{2}$ ort' to '60 fl. weniger $1\frac{1}{2}$ ort'.

The following two questions may be noticed because juxtaposition of the fractions denotes addition: they are consecutive questions, both on D 6′.

"Item einer kauft 10 lb pfeffer vmb 8 fl $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{5}$ $\frac{1}{6}$ ains fl. wie theuer kombt $1\frac{3}{4}$ lb. Facit 1 fl 4 ss 15 ϑ 1 h. $\frac{4}{5}$" and "Item einer kaufft 1 c. alaun für 5 fl $\frac{3}{4}$ $\frac{2}{3}$ $\frac{1}{16}$ eins fl. wie theur komē 4 c. $\frac{1}{2}$. Facit 29 fl. 1 ss 7 ϑ 1 heller".

The occurrence in Peer of Widman's *Regula fusti* question (about the purchase of 27 ct 81 lb of cloves) has been already mentioned in the note to § 65 (Böschensteyn).

Apianus (1527), § 71.

§ 71. As mentioned in § 54 Apianus gives a question like Widman's fig question in which + and − are used and defined.

This question, which occurs on E vi′, is: "Item einer kaufft 9 Lagel öl wegen wie hernach volgt. Kost 1 ct lauter 14 fl 3 ss 14∂, vnd thara vor das holtz an jtlichem Vass abgeschlagen 19 lb. Ist die frag was ist das öll wert, gerechent auff Schwarze Müntz.

Wegenn.

die erst	ct lb		ct lb		ct lb
Lagel	3 + 38	Vierdt	4 + 16	Sybēt	3 + 6
Ander	4 + 44	Fünfft	3 + 16	Acht	5 – 12
Dritt	4 – 10	Sechst	2 + 36	Neūd	5 – 31

"Das + bedeut mehr, das – bedeut minder Facit Sūma ct 34 lb 10. Multiplicir 9 lagel mit 19 lb thara. Facit 1 ct 71 lb Subtrahir von 34 ct 10 lb, bleiben 33 ct 6 pfundt Machs nach der Regel. Facit 479 fl 1 ss $14\frac{16}{25}$∂".*

In the following questions, which occur further on in the book, minder is used.

(G i′). "Item einer kaufft ein vass Honig, wigt $3\frac{1}{4}$ ct minder 5 lb, schlecht ab thara für das holtz 8 lb..."

(G ii). "Item 1 stumpff Saffran wigt $56\frac{3}{4}$ lb vnd 10 lb thara für den stūpff, kosten $4\frac{1}{4}$ lb 18 fl minder anderhalb orth".

(G iii). "Item einer kaufft $12\frac{1}{2}$ vnd $\frac{3}{4}$ lb vn̄ $\frac{1}{4}$ von $\frac{3}{5}$ auss $\frac{1}{3}$ lb kosten $3\frac{1}{2}$ fl minder $\frac{1}{5}$ vō $\frac{3}{4}$ auss $\frac{1}{9}$ eins fl...".

There is also a question in which minus occurs, viz.

(M i). Itē einer kaufft 1 stumpff Saffran, wigt drithalben centē minus 6 lb, thara für den stumpff abgeschlagen drithalb pfūdt. Helt 1 ct fusti 20 lb...".

Rudolff (1530), § 72.

§ 72. In 1530 Rudolff published an *Exempel Büchlin*, consisting of a number of arithmetical questions, all of a commercial character, with some explanations. The title-page is reproduced on p. 159 of *Rara Arithmetica*, where it is stated that the book consists of 292 problems. I have not seen the original edition, but I presume that the *Exempel*

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* There are errors in this solution for the sum of the weights as given is 34 ct 3 lb and subtracting the tara of 1 ct 71 lb the net weight is 32 ct 32 lb. Apianus's values are 34 ct 10 lb and 33 ct 6 lb, which are not consistent with each other, as they do not differ by 1 ct 71 lb. The value in money, 479 fl 1 ss $14\frac{16}{25}$d, is correctly derived from the net weight 33 ct 6 lb (taking 7 shillings to the florin and 30 pence to the shilling). The question and solution are unaltered in the Frankfort edition of 1537; but the errors are corrected in that of 1544, where the weights are given as above and the value in money as 466 fl 3 ss $26\frac{13}{25}$d, which is correct.

Büchle, which is appended to the 1561 edition of the *Künstliche Rechnung** is a reprint, and it is from this edition that I quote.

Among these questions I have noted the following uses of minus. The number prefixed is that of the question.† (95) '18 fl. minus $\frac{1}{2}$ ort'; (107) '7 ceñ. minus 12 lb.'; (120) '12 fl. minus 1 orth.'; (125) '3 ceñ minus 5 lb'; (126) '2 fl minus $\frac{1}{2}$ ort.'; (127) 'eymer 2. Ime 9 minus 2 mass'; '2 Eymer, minus $2\frac{1}{2}$. Ime'; and '2 fl minus 3 ss.' [an eymer is 16 Ime and an Ime is 4 mass]; (157) '17 eimer, minus $\frac{1}{8}$.'; (165) '2 ceñ minus 6 lb.'; (167) '23 ceñ minus 3lb'; '29 ceñ minus 4 lb.'; (258) '1 neuntel, minus 1 Kukis.' [a kukis is $\frac{1}{16}$ of a neuntel]; (271) '10 fl minus $\frac{1}{2}$ ort'.

Two questions seem worth quoting in full.

(109) "Vier trümer Samat, halten eln $3\frac{1}{4}$ $4\frac{2}{3}.2\frac{1}{5}.3\frac{1}{2}$.je $\frac{2}{3}$ aus $7\frac{3}{4}$ Eln, pro 19 fl. $\frac{1}{3}$ vnd $\frac{1}{4}$ eins fl. minus $\frac{1}{2}$ von $\frac{3}{4}$ aus $10\frac{8}{9}$ fl. Wieuil ist der Samat aller werd, auch wie theur die Eln".

This question, which is clearly merely an exercise in fractions, is, that if $\frac{31}{6}$ of an ell costs $(19+\frac{1}{3}+\frac{1}{4}-\frac{1}{2}\times\frac{3}{4}\times\frac{98}{9})$ fl, that is $\frac{31}{2}$ florins, what do $13\frac{37}{60}$ ells cost? ‡

(196) "Einer hat Eyer kaufft, gibt mir solchen bericht, das 21 eyer, vnd 2 Putschändel, gelten 14 Wiener ϑ minus $4\frac{1}{2}$ ayr. Ist nun ferner die frag, wieuil eyr pro 1 kreutz. kommen, gelten 3 Putschändel 1 kreutzer, oder 4 Wiener ϑ. Facit 9 ayr pro 1 kreutzer".§

This question is of the same type as Widman's egg question: 2 Putschändel are equal to $2\frac{2}{3}$ Wiener ϑ, and therefore the question is equivalent to 21 eyer + $2\frac{2}{3}$ϑ = 14ϑ − $4\frac{1}{2}$ eyer, whence $25\frac{1}{2}$ eyer are worth $11\frac{1}{3}$ϑ, and 1 ey cost $\frac{4}{9}$ of 1ϑ, that is, $\frac{1}{9}$ of a kreutzer.

The word weniger is used in (34) "Ein schock Hüner weniger 3, das par pro 17ϑ. vnd ist 60. 1 schock. Facit 2 fl 0 ss 4ϑ 1 hr."

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* "Küntsliche rechnung mit der ziffer vnd mit den zalpfenningen sampt der Wellischen Practica, vnd allerley vortheil auff die Regel De Tri. Item vergleichung mancherley Gewicht, Elnmass, Müntz, &c. Auff etliche Land vnd Stett. Gemehret mit 293. Exempeln, von mancherley Kauffhendeln, mit erklärung, wie dieselben zumachen vnd inn die Regel zu setzen sein. Auffs new widerumb fleissig vbersehen, vnd an vil orten gebessert. Alles durch Christoffen Rudolff in Wien verfertiget. 1561". The colophon is "Getruckt zu Nürmberg, durch Christoff Heussler. 1561".

† In the 1561 edition (which is the only one I have seen) the questions are numbered, and I refer to them by their numbers.

‡ The question occurs under the title "Exempel zu sonderer vbung der Species in Brüchen".

§ It is stated in question 179 that a Putschändel is a white Bohemian penny.

Albert (1541), § 73.

§ 73. In 11° (Albert, 1541) there are a number of questions in which weights and amounts of money are expressed by differences. The first use of the sign is on D viii′ in the question "Item 1 Thonne Putter vmb 15 fl 9 gr, Wie ein pfund? Facit 1 gr 6℈ Helt 2 centner—4 pfund". Here the barrel holds 216 lb of butter and the question is to find a fourth proportional to 216, 324, 1.

The next question involving —, which occurs on I iii, is "Item 12 cent. 4 stein—6 pfund Talg will einer verkeuffen, den cent vmb 4 fl 12 gr, Wieuiel ist die Sūma". In a question on I vi the weights are '3 cent.—7 pfund' and '4 centner 19 pfund'; and in another on I vi′ they are '2 cent. 16 pfund', '3 cent.—7 pfund', '4 cent. 35 pfund'. In a question on I vii′, in which the weights do not involve the sign —, the cost of a centner is '7—½ fl'. In other questions on I vii′, I viii, I viii′ the weights '6 ct—13½ lb', '4 ct—5⅓ pfund', '2 cent—9 lb ¾' occur in conjunction with weights in which the lbs are additive, *i.e.* in which no sign is used.

The negative sign occurs in connection with money in '65 fl—3 fl' (K vii), '3 fl—1 ort' (L i), '3 fl—½ ort' (L i), '7 fl—¼' (L i′), '9 fl—3½ ort' (L iii′), '8 fl 3 ort—7 heller' (L iii′), '9 fl—⅓' (L v′), '17 gr ½—2℈' (M iv), '5—¼ fl' (M vii′).

In several questions the word mehr is used for vnd, *e.g.* on K ii, where a person buys "2721 schwartze Seelandische Zmaschen...Mehr, 3603 Denische Lambsel...Mehr, 399 Denische Schörling..." and on K iii where a question begins with Mehr, viz. "Mehr kaufft er, 918...Marder...Mehr 1139 Lasset..."

Spenlin (1546), § 74.

§ 74. In 14° (Spenlin, 1546) there are three questions in which minder is used in expressing money or weights: thus on p. xlvii′ a price is given as '9. fl minder 2½ orth', on p. xlviii the weights of 5 barrels of oil are given as '3.cent 3.lb', '2. cent 54.lb', '4.cent minder 11.lb', '2.cent 15.lb', and '3.cent 28.lb': and on the next page 3 'thunnen' of honey weigh '1. cent 74. lb.', '2. cent 21. lb', and '3. cent minder 6. lb.'

Riese (1550), § 75.

§ 75. In 15° (Riese's *Rechenung nach der lenge*, 1550) there are a great number of questions of the same kind as those which have just been described under Rudolff, in which minus generally occurs, though weniger is sometimes used.

In question 118 (pp. 23′ and 83′) two sacks of pepper weigh "3 ct minus 17 lb, vnd 2 ct 19 lb, tara fur die seck 5 lb..." and in describing the working on p. 23′ Riese says "Summir, Mach ct zu pfunt, Nim ab das minus vnd tara..." *i.e.* he sums the positive lb, which amount to 519 lb, and subtracts the sum of the negative lb and the tara, which comes to 22 lb, leaving 497 lb as the weight. This is his usual procedure, and the words 'Nim ab das tara vnd minus' or 'Nim ab das minus vnd tara' occur several times, *e.g.* in questions 122, 123, 124, 125, 130.

In question 119 (pp. 24 and 83′) he gives the cost of 1 lb of saffron as '4 fl minus $1\frac{1}{2}$ ort' on p. 24, and as '4 fl weniger $1\frac{1}{2}$ ort' on p. 83′; and in question 128 the cost is '4 fl minus $\frac{1}{2}$ ort' on p. 25, and '4 fl weniger $\frac{1}{2}$ ort' on p. 84′. These are the first questions in which minus or weniger occurs in connection with money. Minus is also used to connect money in question 123, and it connects centners and lbs in questions 118, 122, 123, 124, 125, 127, 130, 131, 132, 173. In questions 131 and 132 it occurs in two of the weights. In only one question (123) is it used both for weight and money (as in Widman's pepper question).

On p. 139 a weight is given as '3 ct minus 11 lb', and on p. 163′ as '11 ct minus 4 lb'; and on pp. 137′ and 138 prices are given as '3 fl minus $\frac{1}{2}$ ort' and '2 fl minus $\frac{1}{2}$ ort'.

The word minus or the sign — is also used to connect quantities of the same denomination as *e.g.* in '$8\frac{1}{4}$ minus $7\frac{1}{2}$' (p. 153), '71 fl − 3' (p. 158′), '40 − 12', &c. (p. 159), and the word weniger is used in a similar way on p. 160.

The Algorismus of Ambrosius Lacher de Merspurg (1506–1510), §§ 76–78.

§ 76. After I had written the preceding account of the sixteen books, whose titles were given in § 42, I met with a German Arithmetic containing complicated commercial questions of the same type as in Widman, which, though undated, was certainly published before 1510 (see § 77). It will be seen that some of the questions involve minus in the data.

The title of the book is "Algorithmus Mercatorum varijs propositionibus eorūdem cōtractibus inseruientibus multū decoratus. per magistrum Ambrosium Lacher de Merspurg. Mathematicum in Studio nouo Franckfordiano".* It consists

* Below the title is a large coat of arms, which is that of Dietrich von Bülow, Bishop of Lebus (near Frankfort a. O). He was appointed Bishop of Lebus in 1490 and participated in the foundation of the University of Frankfort in 1506, of which he was the first Chancellor.

of 14 quarto leaves Ai–Avi, B1–B4, Ci–Civ, and is printed in double columns. The first seven pages and part of the eighth (Ai′–Av) are a reprint of Peurbach's Algorismus (§ 126) up to the end of *De Divisione.* Lacher then proceeds "Jam sequitur pulcra et multum vtilis applicatio predictorum ad varium mercatorum vsum per multas propositiones huc accomodatas". Then follows the paragraph about the rule of three, which appears sometimes as an addition to Peurbach's Algorismus and sometimes as a part of it. After this comes more explanation of the rule, followed by a short account of fractions (which were not included in Peurbach's Algorismus), and then he at once proceeds to give 18 examples with their results.

The fifth of these examples is: if a cantarus of wine costs 5 denarii how much can be bought for 3 florins minus 3 denarii (3 fl min⁹ 3 dn̄)? The ninth example also involves money expressed by means of minus. This question is: a person buys 525 oxen, paying for each pair 9 fl minus 8 gr (dās sēper pro duobus paribus 9 fl min⁹ 8 gr): his expenses are 158 fl 18 gr and he wishes to gain 268 fl, what must the price of each ox be?

The fifteenth question is formed exactly on the model of Widman's fig question (§ 11), except that the words plus and minus are used instead of + and −. This question is: "Itē quidā emit 6 lagēas fic⁹ dās pro quolibet ct 4 fl pōderatque vt ordo edocebit subscript⁹, queritur de sūma tocius.

Lagn̄	ct		lb
Prima	4	plus	21
Secūda	4	min⁹	30
Tertia	3	plus	56
Quarta	5	min⁹	12
Quinta	4	min⁹	8
Sexta	5	min⁹	38

quelibet aūt lag. ī ligno pōderat 25 lb ft 93 fl 17 gr 7 dn̄ $\frac{9}{25}$". The sum of the lagenæ is 2489 lb, and subtracting 150 lb for the wood there is left 2339 lb. This at 4 fl the ct amounts to 93 fl 17 gr 17$\frac{9}{25}$d, which is the result given. This question, like Widman's, relates to figs, and there is a fixed deduction for the weight of each lagena.

In the next question a person buys 3 baskets of raisins (3 corbulas pastarum vuarum), of which the first weighs 1 centner, the second 4 stein minus 6 lb (4 lap. mi⁹ 6 lb.), and the

third '1 ct minus 10 lb': he pays 8 fl 24 gr for each centner, and 14 lb is deducted (defalcātur) for the weight of the baskets. Here, as in other places where a stein occurs (as *e.g.* in § 56), the centner is 110 lb and the stein 22 lb, so that the net weight, after deducting the weight of the baskets, is 278 lb. This at $8\frac{3}{4}$ fl the centner gives 22 fl 3 gr $5\frac{1}{11}$d, which is Lacher's result.

In the next question a weight is given as '13 ct plus 18 lb'. In the next and last example of the series a person buys 3 lagenæ of olive oil, of which the first weighs '2 ct pl⁹ 18 lb.', the second '3ct min⁹ 33lb', and the third '3 ct & 4 lb', from which 20 lb in each centner is to be deducted for the weight of the jars (resecātur de quolibet cētenario. 20 lb ratiōe vasis), and 1 ct costs 18 fl. The result in this question shows that the centner is to be 110 lb, so that the lagenæ weigh 869 lb, from which $\frac{2}{11}$ is to be deducted as the weight of the jars, leaving 711 lb. This at 18 fl for 110 lb gives $116\frac{19}{55}$ fl, which is Lacher's result. This question closely resembles Widman's oil question (§ 18, p. 12), in which the weights were '2 ct 18 lb', '3 ct—32 lb', and '3 ct + 5 lb', and the weight of the wood of the barrels was given as 9 lb on each centner.

After these questions there are tables of weights and measures, and then a number of practical commercial questions relating to the exchange of money, the conversion of weights, partnerships, barter, &c. On C i′ he gives two questions which involve fusti. In the first of these a person buys at Venice 1725 lb of cloves, paying 12 ss for 1 lb of the [pure] cloves and 2 ss for 1 lb of the fusti (fusti vel impuri), each centner containing 15 lb of fusti: in the second, a person buys at Nuremberg 6 ct 95 lb of saffron, paying 340 fl for 1 ct, and each centner contains 20 lb of fusti. The result shows that in this question the fusti is not paid for. In each of these questions the centner is 100 lb.

§ 77. Ambrosius Lacher, of Mersburg on the Lake of Constance, the writer of the *Algorithmus Mercatorum*, began his studies at Leipzig in 1488, and became a Bachelor there in 1490. He matriculated at Wittenberg in 1502 and became Master in 1504. On the foundation of the University of Frankfort on the Oder in 1506 he came there as teacher in mathematical studies and established a private printing press in which he printed the books he required for his lectures. These were: in 1506, an edition of the first four books of Euclid with Campanus's commentary: in 1508 "Arithmetica Muris...nuper bene reuisa lectaque ordinarie atque impressa per Magistrum Ambrosium Lacher de merspurgk. Mathe-

maticum etc. In studio Frankfortiano Anno 1508. die 26 Junij, studij vero prefati anno 3." and "Epytoma Johannis de Muris in Musicam Boecii . . . diligenter reuisa. ordinarie lecta atque impressa in studio nouo Frankfordiano Anno salutis 1508. studii vero prefati 3. in die sancti Galli"; and in 1511 "Tabule Resolute de motibus Planetarum..." The *Algorithmus Mercatorum* has no date, but must have been printed before 1510, for in that year a second edition was published at Leipzig under the title "Algorithmus mercatorum Magistri Ambrosij Lacher de Merspurg Mathematici de integro et fracto numero: ac varijs propositionibus eorundem contractibus inseruientibus. bene emendatus per Baccalaureum Bartholomeum Schoebel In florentissimo Gymnasio Lipssensi ad monetam nostram nuper recalculatus. Impressum Lyptzigk per Baccalaureum Martinum Herbipolensem Anno domini 1510." Thus the original work must have been printed before 1510, and it could not have been printed before 1506, when Lacher set up his press.

Lacher was rector of the University of Frankfort a. O. in 1516. He then studied medicine, and became Bachelor and Licentiate of Medicine in 1522, but he remained in the Arts' faculty till his death in 1540.*

§ 78. The similarity of the questions in the *Algorithmus* to those of Widman is explained by the fact that Lacher was at Leipzig in 1488-89 when Widman was lecturing and when his *Rechenung* was published. There can, I think, be no doubt that for questions such as those relating to figs, oil, fusti, &c., and probably for others, Lacher was indebted to Widman's teaching or to his book. It is interesting to find these commercial questions appended to Peurbach's *Algorismus* and a subject of University instruction. Lacher wrote in Latin, and I have met with no other strictly mercantile Arithmetic of this period which is in Latin. He does not mention Widman;

* The facts of Lacher's life, as well as the titles of the books printed by him (except the *Algorithmus*) and of the second edition of the *Algorithmus*, have been taken from a paper by Dr. Gustav Bauch, "Drucke von Frankfurt a. O." in vol. xv. (1898) of the *Centralblatt für Bibliothekwesen* (pp. 241–260). Bauch gives a list of all the books printed at Frankfort a. O. up to 1528, with an account of the various printers and documents relating to them. Of Lacher's printing he says that it sometimes gives the impression of amateur work. Lacher's books seem to be very scarce. Bauch states that there is a copy of the *Algorithmus* in the Royal Library of Berlin, and of the Leipzig edition in the Hof u. Staats Library at Munich. The copy which I have used is in the Library of Trinity College, Cambridge. Lacher is not mentioned in *Rara Arithmetica*. Panzer and Kästner mention the Euclid. The latter gives an account of it on pp. 302–304 of vol. i. of his *Geschichte der Mathematik* (Göttingen, 1796). He there quotes Lacher's address to the reader, which begins "Ambrosius Lacher de Merspurgk Constanci diocesis Arcium liberalium magister sacreque Mathematice studii nostri ordinarius candido lectori".

nor do any of the writers of the sixteen books described in §§ 41–75, excepting only Riese, in the dedication of his manuscript Coss (§ 49). The *Algorithmus* affords no evidence of Lacher's having been acquainted with algebra. He used the words plus and minus, but then he wrote in Latin: still he might have used et in place of plus. I do not think that any inference can be drawn from his not using + and −. He may have thought these signs unnecessary in an Arithmetic, or he may not have had access to the necessary type. In the *Algorithmus* a number of initial letters at the beginning of paragraphs are missing, although space is left for them: and this was presumably for want of type.*

The signs + and −, § 79.

§ 79. The signs + and − are freely used in all the Algebras, viz. in 4° (Grammateus, 1518); 7° (Rudolff, 1525); 12° (Stifel, *Ar. Int.* 1544); 13° (Stifel, *Deut. Ar.* 1545); and also in Riese's manuscript Algebra (1524).

In 4° (Grammateus, 1518) they are defined as vnnd and mynnder; in 7° (Rudolff, 1525) as plus and minus; in 13° (Stifel, *Deut. Ar.* 1545) as 'vnd oder mer', and 'weniger oder minder'. They are not defined in 12° (Stifel, 1544), on their first occurrence in the book, nor in Riese's manuscript Algebra.

In the Arithmetics the signs + and − are used in the rule of false whenever that rule is given, viz. in the arithmetical portion of 4° (Grammateus, 1518); in 6° (Riese, 1525), where they are called plus and minus; in 10° (Apianus, 1527), where − is called minus, but no word is given as the equivalent of +; in 11° (Albert, 1541), where they are called mehr and weniger; in 14° (Spenlin, 1546), where no equivalent words are given; and in 15° (Riese, 1550), where they are called plus and minus. In 5° (Grammateus, 1521), 8° (Rudolff, *Arithmetic*, 1526), 9° (Peer, 1527) they are not used at all. Except in the rule of false they are not used in the arithmetical portion of 4° (Grammateus, 1518) or 6° (Riese, 1525). In Apianus they are used only in the question which is almost exactly copied from Widman's fig question (§ 71). In 11° (Albert, 1541) the sign − is freely used. In 12° (Stifel, *Ar. Int.*, 1544) the words plus and minus are used throughout in the arithmetic (lib. i) and in the rule of false (appendix to lib. i) excepting in the one case 1, 3 − 1, 3, 3 + 1, &c., mentioned in § 57, but + and − are freely used in the algebra (lib. ii and iii).

* The omission of initial letters in Lacher's Euclid is mentioned by Kästner in his account of it referred to in the preceding note.

The words minus and plus, § 80.

§ 80. In the Arithmetics the word minus (besides its use in the rule of false) was generally used, as also were the words minder and weniger. We may suppose that its meaning was well known, as it is never explained. It occurs very early in the arithmetical portion of 4° (Grammateus, 1518) in the explanation of multiplication (§ 46). It does not occur in 5° (Grammateus, 1521), but it is used in 6° (Riese, 1525) and in 8° (Rudolff, 1526). In 10° (Apianus, 1527) minus occurs once. In 11° (Albert, 1541) the sign – and not the word minus is used. In 14° (Spenlin, 1546) minus is not used. In 15° (Riese, 1550) it is freely used throughout.

Apart from the rule of false, plus is not used in the Arithmetics except in 15° (Riese, 1550), where it occurs twice (§ 61).

Widman in his fig question defined the sign – as minus, but he does not use the word elsewhere except in the rule of false, which is the only place where he uses plus. Lacher, who wrote in Latin, substituted the words plus and minus for Widman's signs + and – (§ 76).

It is not surprising that the word minus occurs often and the word plus scarcely at all (except in the rule of false), for minus is really needed as a technical word for 'diminished by' (in preference to minder or weniger), while und or simple juxtaposition suffices for addition.

The use of + *and* – *in algebra and arithmetic*, § 81.

§ 81. In algebra cossic symbols, irrational quantities, and natural numbers have to be combined by addition and subtraction, and it is evident that the special signs + and – are more suitable for this purpose than words or the mere abbreviations of words, even if consisting of a single letter, and that the 'rules of signs' in addition, subtraction, multiplication, and division are more conveniently expressed by signs than in words.

In arithmetic there is not the same need for the signs, for the quantities to be connected by addition and subtraction are of the same kind, and therefore the operations indicated by them can be performed, but in algebra quantities have to be united to form a single expression, although they cannot be combined into a single term.

It is easy to see therefore why the signs + and – were found so useful and even necessary in algebra, and yet were little used in arithmetic.

In considering the prevalence of the signs the printer also should be taken into account, for although he always had the type for –, a special type would have to be made for +, unless the alternative of a clumsy substitute, consisting of two –'s with a vertical or sloping line between, was accepted.

The questions involving – or minus in the data, §§ 82–83.

§ 82. Coming now to the examples of the type of Widman's fig and pepper questions (*i.e.* in which weights or denominations of money in the data are connected by signs or by minus or an equivalent word) the account which has been given in §§64–75 shows that there were three such examples in 6° (Riese, 1525) and numerous examples in 8° (Rudolff, 1526), 9° (Peer, 1527), 10° (Apianus, 1527), (Rudolff, 1530) described in §72, 11° (Albert, 1541), 15° (Riese, 1550). Except in the one question in 10° (Apianus), quoted in § 71, the sign + is never used, and it is only in this question and in 11° (Albert, 1541) that – is used. In 9° (Peer, 1527) the words minder and weniger alone are used, and in the three examples in 14° (Spenlin, 1546) minder is used; but in all the other works minus occurs, though weniger and minder were also used. As will be seen in §76, questions like Widman's were given by Lacher (1506–1510), but + and – were not used.

§ 83. There is however nothing to show definitely whether these questions represented actual commercial transactions or were merely constructed as arithmetical exercises on the model of Widman's questions. It is of course quite possible that weights were recorded as centners diminished by a certain number of lbs as well as centners with a certain number of lbs added: but on the other hand it is quite possible that Widman in his exuberance over the signs + and – constructed this kind of question to display their utility and the method of treating them; and this is rendered more likely by his use of the signs in money where such a mode of expression could scarcely have arisen naturally. A few of the questions evidently were intended merely as exercises in fractions and the use of – or an equivalent word.

In the weights the word minus or its equivalent occurs only in connection with centners and lbs, or centners, steins, and lbs, and it was always applied to the lbs. These were presumably the only weights used for heavy goods. Lacher, who was a close follower of Widman (see §76), uses minus with centners, steins, and lbs: in the sixteen books it was only so used by Albert.

I do not know of any questions in Borgi (1484) or Paciolo* (1494) in which a minus or its equivalent occurs in the weights or other data, nor have I met with such a question in any of the later Italian Arithmetics referred to in this paper (§§109–123).

It would seem that questions of this type are peculiar to Widman and his German successors, who may all have derived their inspiration directly or indirectly from him. Venice was so important a commercial centre that it is difficult to think that the Venetian Arithmetics would not have contained examples of this mode of expressing weights or money if it was in actual use in commerce.

The rule of false, §§ 84–87.

§ 84. In all the books in which the rule of false is given, with the sole exception of 12° (Stifel, *Ar. Int.* 1544), the signs + and – are used to denote whether the error is in excess or defect. These books are 4° (Grammateus, 1518), 6° (Riese, 1525), 10° (Apianus, 1527), 11° (Albert, 1541), 14° (Spenlin, 1546), 15° (Riese, 1550).

It is interesting to notice the form of words in which the signs are introduced, and whether plus and minus are used. Grammateus (1518) merely says "Ist zu viel setze + Ist aber zu wenig setz –". Later on, in the algebra, he explains "+ ist vnnd, – mynnder" (§ 45). Riese (1525) uses the words plus and minus: "zuuil so bezeychenn sie mit dem zeychen + plus wu aber zu wenigk so beschreib sie mit dem zeychen ÷ minus genant" (§ 48). Apianus (1527) uses the word minus but not plus "zü zeitten meher oder minder, was zü vil ist vermerck mit dem zeichen plus +. Ist zü wenig, mit dem zeichenn minus –" (§ 55). Albert (1541) does not use the words plus or minus: "So mach von stund, nach der Falschen zal, diese linie mit durchgezognem stricklein, also —/— Vnd bedeutet Mehr . . . Wo aber zu wenig, zeuch nach der Falschen zal diese linie — Vnd bedeut weniger" (§ 56). Spenlin does not use the words plus and minus: "bedeut das zaichen + züuil, vnd das ÷ zü wenig" (§ 60). Riese (1550) defines the signs as plus and minus "mit dem zeichen + das ist plus . . . mit dem zeichen – das ist minus" (§ 61).†

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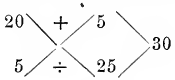
* The works of Borgi and Paciolo are referred to on p. 4.

† Widman had no occasion to define the signs + and –, as he had used them freely in the previous portion of his book. Although they are prefixed to the errors in the diagrammatic representations of the positions and errors (with the cross), he uses the words plus and minus in the explanations.

§ 85. Widman placed the error to the right of the position to which it belonged, the second position being under the first position and the second error under the first error, and he connected the first position with the second error, and the second position with the first error, by a cross, but Grammateus (1518), Riese (1525), and Albert (1541), though they followed Widman in the mode of placing the positions and errors, dispensed with the cross. Apianus, however, wrote the error under its position (prefixing its sign) and used the cross. In Spenlin (1546) the cross was made very large, the arm reaching to the side of the position and error, and the difference was indicated by guide lines: thus



§ 86. In the rule of false the signs + and − do not indicate addition and subtraction: they merely qualify the numbers to which they are prefixed, so that they may be suitable for treatment by the rule which is to give the correct result. When therefore the error follows the position in the same line we must suppose the insertion of some word such as 'gives' or 'produces' (which might now be indicated by putting a comma after the position). The signs merely replaced the plus and minus or piu and meno of the earlier writers, which indicated that the result given by the position was more than the truth or less than it, the actual deviation from the truth being the error, *i.e.* that the result was more than the truth by the error, or less by it. It seems strange that when + and − were being used as signs of addition and subtraction they should be placed with another meaning between the position and the error*. When the cross lines were inserted the whole arrangement could be regarded as a diagram shewing the positions and errors, and which of them were to be multiplied, but when the cross was omitted there was even less justification for placing the position and error in the same line as if forming a binomial expression.†

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* Thus for example (on E vii) Grammateus writes

900 + 62

600 − 292

the positions being 900 and 600 and the corresponding errors + 62 and − 292.

† It is noteworthy that as soon as the signs + and − were introduced they should have been so generally used in the rule of false. The words plus or minus, or some equivalents, were required, and, as has been said, the 'law of signs', which was required in the subtraction and multiplication, is more readily expressed by + and −.

§ 87. Among the Italian writers, Leonardo Pisano* (1202) placed the error under the position and used the words plus and minus, which he wrote over the error (*i.e.* between the position and its error). In Borgi (1484) the error is placed under the position with piu or men on the outside of the error in the same line with it. In Paciolo (1494) the error is placed under the position, the abbreviations p̄ or m̄ being inserted in the right and left angles of the crossing guide lines close to the point of intersection. Leonardo wrote in Latin and Borgi and Paciolo in Italian. All three placed the error under its position and used guide lines to indicate which numbers were to be multiplied. These formed a simple cross if the position for only one quantity was shown in the diagram, but if there were several quantities involved in the data, and the positions for the first and also for those derived from it were shown in the diagram, the guide lines connecting the errors with the opposite positions formed a more complicated figure of intersecting lines. In the latter case (*i.e.* when the positions, not only for one quantity, but for those dependent on it were shown) the error necessarily had to be placed below its position. In Leonardo and Paciolo the position for only one quantity is included in the final scheme, so that the guide lines form a simple cross, but in Borgi they form a system of intersecting lines. In later Italian books (of the first half of the sixteenth century) the error is placed sometimes below and sometimes to the right of the position, and the crossed lines are sometimes dispensed with. The abbreviations p̄ and m̄ are frequently used, and sometimes the word per is prefixed to the position. The signs + and – are not used in Italian books during this period.†

The use of the words tara and fusti by German writers, §§ 88–102.

Introductory remarks, § 88.

§ 88. While examining the use of the words plus and minus and their equivalent signs, I became interested in the words tara and fusti. Both words came to the German writers on arithmetic from Italy. The former is derived

* "Scritti di Leonardo Pisano . . . pubblicati da Baldassare Boncompagni" (Rome, 1857), vol. i, p. 319.

† Two fifteenth-century Latin manuscripts on the rule of false in the Hof u. Staats Library at Munich were printed by Curtze in vol. xl. of the *Zeitschrift für Math. u Phys.*, Supp. pp. 35–49, in both of which the words plus and minus are used in the diagrams. In the first the error is placed under the position, the word plus or minus being placed outside the error in the same line. In this

from the Arabic word tarha, meaning 'what is thrown away', and the latter is the plural of fusto, the stem of a plant*. The general use of the words among the German writers is that tara denotes the weight of the box, barrel, sack, etc., in which goods are packed, and that by fusti is meant the part of the goods which is of inferior value, such as the stalks which accompany cloves. Thus in general the tara is an amount for which no payment is made, and fusti is paid for at a much lower rate than the principal part of the goods.

Widman (1489), § 89.

§ 89. Widman uses the word fusti but not tara. In his fig question, in which + and − first occur (§ 11), he subtracts the weight of the three barrels (Nu solt du fur holcz abschlahn̄ albeg fur eyn lagel 24 lb), and in the pepper question which immediately follows it (§16) he subtracts $3\frac{3}{4}$ lb for the weight of the sack (vn̄ sol fur den sack abschlahn̄ 3 lb + $\frac{3}{4}$). In a question relating to grapes† 29 lb is taken off for the wood (vn̄ geth ab an dē 3 lageln̄ fur das holcz 29 lb). In the next question on oil (§ 18) 9 lb is taken off in each centner for the wood (Vnd get ab fur das holcz ye fur 1 ct 9 lb), and in the question relating to soap (§ 18) 12 lb in the centner is taken off for the wood (vnd geth ab fur das holcz ye fur 1 ct 12 lb).

It will be noticed in these three questions of Widman's that there are two kinds of deduction from the gross weight, viz. (1) the actual weight of the containing vessels (barrels, sacks, etc.), and (2) a percentage on the gross weight to represent the weight of these vessels or other allowance. Widman does not use the word tara, but both of these kinds of deductions were so denoted by subsequent German writers.

Under *Regula Fusti* Widman gives two examples. In the first, 2781 lb of cloves are bought, the pure cloves costing 11 ss 3 hlr the lb, and the stalks (fusti) 2 ss − 3 hlr the lb, and

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manuscript the positions of all the quantities in the data are shown, and the guide lines form a system of intersecting lines. In the second manuscript the position for only one quantity is shown, the position and error are written in the same line, and a cross is used. In the first manuscript, in the description of the rule, the excess and defect of the result are called 'abundancia seu superfluum' and 'defectus', and in one diagram, where both results are in excess of the truth, superfluum is used instead of plus. Plus and minus seem to have been generally used in the diagrams in the rule of false, although they were rarely used as signs of addition and subtraction.

* Drobisch on p. 19 of his work *De Ioannis Widmanni...compendio*, referred to in § 1, writes "Vocabulum fusti fucum vel futilia in merce significat", and he puts in a footnote "Ab Italica voce fusti, proprie cauliculum baccarum uvæ passæ significans".

† Widman, p. 93.

one centner contains 13 lb of fusti "Eyner kaufft 278 [should be 2781] lb ye 1 lb lautter pro 11 ss 3 hlr vnd 1 lb fusti pro 2 ss – 3 hlr Nu helt ye 1 ct 13 lb fusti". This is followed by a question on saffron in which 100 lb cost $94\frac{1}{3}$ fl, and the cost of $384\frac{1}{4}$ lb is required "vnnd 1 ct helt 15 lb vnreyn". Here the 'vnreyn' amounts to $57\frac{51}{80}$ lb and the solution shows that only $326\frac{49}{80}$ lb is paid for, so that the 'vnreyn' is entirely waste.

Thus Widman uses the word fusti when the inferior matter mixed with the goods is of some value and is paid for, and vnreyn when it is valueless and not paid for: but in his explanation of the *Regula Fusti* the words vnreyn and fusti are treated as equivalent.*

Lacher de Merspurg (1506–1510), § 90.

§ 90. In the Algorismus of Lacher, described in § 76, the word tara is not used, which is what we should expect from his close following of Widman. The amount to be subtracted from the gross weight to allow for the weight of the containing vessel is expressed by 'quelibet aūt lag. ī ligno pōderat 25 lb', 'defalcātur 14 lb pōd⁹ corbularum', 'Prima vero lagena pōderat in ligno. 21.lb', 'resecātur de quolibet cētenario. 20 lb ratiōe vasis'.

There are two fusti questions in one of which 1 lb of cloves costs 12 ss and '1 lb fusti vel impuri' 2 ss, a centner containing 15 lb of fusti; and in the other a centner of saffron contains 20 lb of fusti, and a centner of [pure] saffron costs 340 fl. In this question the fusti is waste and is not paid for. These questions correspond exactly to Widman's, in which the inferior matter mixed with cloves is of value, but that mixed with saffron is valueless. Lacher makes no distinction between fusti and impuri, for he uses 'fusti vel impuri' for the stalks, etc., in cloves, and 'fusti' for the refuse in saffron.

Böschensteyn (1514), § 91.

§ 91. As mentioned in § 65, Böschensteyn does not use the word tara, but he gives a question relating to oil in which

* This explanation, though the words are printed without a break, is really in rhyming verse. The first four of the six lines are:

"Regula Fusti drey regel habn̄ wil
lauter vnrein mitsampt des musterss zil
auss dem muster thn den fusti formirn̄
denn darnach vonn lautern subtrahirn".

These verses also occur in Böschensteyn (§ 91) and in Peer (see § 98).

11 lb on the centner is taken off for the wood (schlecht ab für das holtz 11 lb am ct), and 1 centner pure (lauter) is sold for 6 fl $1\frac{1}{2}$ ort. Here the solution shows that '11 lb am ct' does not mean 11 per cent, but that 11 lb in every 111 lb is to be treated as wood.

On C vi, under *Regula Fusti*, he gives Widman's verses, writing them in separate lines as verses,* and also his first question, "Item ainer kaufft 27 ct. 81 lb. negelein ye 1 lb lauters vmb 11 ss. 3 hlr. vnd 1 lb. fusti vmb 21 hlr.", which he solves in the same manner.

Grammateus (1518 and 1521), § 92.

§ 92. Grammateus is the first German writer I have met with who uses the word tara. In the question quoted in § 66, six centners of pepper are bought for 50 fl the centner pure (lauter), and 1 centner holds 5 lb tara (hielt . 1 . centner tara. 5. lb). In the solution the 30 lb tara is subtracted, leaving 570 lb, which is paid for at the rate of 50 fl for 100 lb.

It would seem therefore that tara here means waste, *i.e.* matter mixed with the pepper, and which is of no value.

Riese (1525), §§ 93–94.

§ 93. The first use of tara in 6° (Riese, 1525) is where a 'stumpff Saffran' weighs 38 lb 16 lot, and the tara is 9 lot.† The tara is subtracted from the gross weight, leaving 38 lb 7 lot. In the next question a sack of calamus weighs 48 lb 24 lot, and the tara is 2 lb 16 lot: and there are several other questions of the same kind where the tara is a definite amount to be subtracted.

In the question relating to soap, quoted in § 67, the tara is on the centner (tara vff ein cent: 10 pfundt). The solution shows that in every 110 lb, 10 lb is to be deducted as tara.

On E ii′, under the heading 'Fusti', Riese gives a question in which a sack of cloves which weighs $654\frac{1}{2}$ lb and costs 9 ss‡ a lb in Venice is transported to Nuremberg at an expense of 25 fl. A centner contains 15 lb of fusti, and the cloves (pure) are sold at 16 ss a lb and the fusti at 4 ss a lb; and 10 Venice lbs are equivalent to 6 Nuremberg lbs: the question is whether he gains or loses.

* There are some slight changes in the words.

† This question is quoted in the note to § 67 (p. 48).

‡ This is printed 6 ss in the book, but the result shows that it should have been 9 ss.

There is also another question (on E vii) in which the tara is on the centner. Here, 4 barrels of oil weigh "22 cent: 5 stein 6 pfundt kost ein cent: 9. margk ein ort tara 12 pfundt vff ein cent:" and it is explained that a margk is 48 groschen, a groschen 7 pfennige, a centner 132 pfundt, and a stein 24 pfundt. Thus the question is equivalent to finding the value of

$$3030 \times \frac{132}{144} \times \frac{9\frac{1}{4}}{132}$$

marks, which gives Riese's answer, "194 margk 30 grosch: vnd veirthalben pfen".

§94. The following list relates to the 11 examples on D iiii′—D vi′ in which tara occurs. It gives the description of the goods and the receptacles in which they are contained, and the statement of the tara, exactly as printed in the questions.

D iiii′, ein stumpff Saffran wigt...tara 9 lot
D v, ein Sagk mit kalmass wigt...tara 2 pfund vñ 16 lot
„ zwu Zichen mitt paumwollen wegen...tara 37 pfundt
D v′, ein Sagk mit schaffwoln wigt...ttā 21 pfundt
„ ein Vhass mit weinstein wigt...tara 21 pfundt
„ ein Vhass allaun wigt...tara 23 pfunt
D vi, funff Körb mit veygen wegen...tara vff yeden korb 14 pfundt
„ 5 Vesser mit vnschlet wegenn...tara vff yedes vhass 21 pfunt
„ vier Lagel mit oel wegen...tara vff ein cen. 11 pfundt
D vi′, drey thonnen mit honig wegen...tara vff ein cen: 12 pfundt
„ vier lagel mit Seyfen wegen...tara vff ein cen: 10 pfundt

There is nothing in these questions to indicate what the tara represents: it might be any deduction allowed by the vendor. But there can, I think, be no doubt that the tara is the actual weight of the receptacle which contains the goods. This may be inferred almost with certainty from the similar list in §96 relating to Rudolff's questions, in which, when the receptacle is of wood, it is stated that the tara is for the wood.

Rudolff (1526), §§95–96.

§95. In 7° (Rudolff, 1526) there are a number of questions in which the tara is an absolute quantity, and also in which it is 'auff den centner'. Thus in one of the questions quoted in §68 we have "1 stumpf saffran wigt 37 lb $\frac{1}{2}$. thara 9 lot fur den stumpf", and there are other questions in which there is 'thara auff den cen̄'.

In three questions both fusti and tara occur, viz. on I 5′, "Item 1 sack neglein wigt 6 ceñ 54 lb. thara fur den sack 4 lb. helt 1 ceñ. fusti 15 lb, cost 1 lb lauter 16 ss. vnd 1 lb fusti 4 ss"; on I 7′, "Item ainer kaufft 3 seck mit neglein...Thara fur die seck 6 lb, helt der centñ 12 lb fusti, cost 1 lb lauter 6 ss vnnd 1 lb fusti 3 ss 20 ℈"; and on K 7′, '3 seck neglein' contain respectively 8 lb, 10 lb, and 12 lb of fusti per centner, the 'thara' for each sack is 2 lb, and 2 lb 'lauter' costs 1 ducat, and 5 lb of fusti costs 1 ducat. On K 6 there is a question in which a sack of cloves holds 15 lb fusti, and 1 lb 'lauter' costs 16 ss and 1 lb of fusti 4 ss: but there is no thara.

§ 96. The following list is similar to that derived from Riese's questions which was given in § 94. All the questions occur on I 4′–I 5′.

I 4′, 1 stumpf saffran wigt...thara 9 lot fur den stumpf
,, 4 seckh mit mandeln wegen...thara fur dye seck. 12 lb.
,, 3 schaf smaltz wegen...Thara fur holtz 12 lb $\frac{1}{3}$
I 5, 1 sack mit schafwoll wigt...thara fur den sack 19 lb $\frac{1}{2}$.
,, 3 zichen mit baumwoll wegen...thara fur dye sack 37 lb
,, 4 korb mit feigen, wegen...thara fur jeden korb 13 lb
,, 1 vass mit weinstain wigt...thara fur holtz 27 lb $\frac{1}{2}$.
,, 1 vass alaun, wigt...thara fur holtz auf den ceñ. 6 lb
,, 5 vesser mit vnslit, wegen...thara fur jedes vass 21 lb
,, 4 lagel baum oll wegen...thara fur holtz auf 1 ceñ. 11 lb.
,, 3 truchen mit saiffen, wegen...thara fur holtz 10 lb auf 1 ceñ.
I 5′, 3 tonnen honig wegen...thara fur holtz auf 1 ceñ. 9 lb.
,, 3 tonnen honig wegen...thara auf den ceñ 10 lb.
,, 1 sack neglein wigt...thara fur den sack 4 lb.

Whenever the goods are contained in a wooden vessel Rudolff has added that the tara is for the wood: thus the tara is for the wood in the case of the 3 tubs of grease, the vessels containing tartar or alum, the barrels of olive oil, the chests of soap, the casks of honey; and we may assume therefore that when it is for the sack, basket, etc., it is for the weight of these receptacles, and has no reference to the goods themselves.

Other questions in which 'thara' is used occur on I 6, I 7′, K 7, K 7′, L 5′, M 2′, and it is always 'auff den centner', or for the 'sack', 'holtz', or 'jedes vass'. I do not think there is any instance in which 'thara' occurs without the addition of the receptacle to which it relates.

Peer (1527), §§ 97–98.

§ 97. There are similar questions in 8° (Peer, 1527), in which tara is either a definite quantity or a proportion of the

gross weight; and, like Rudolff, in giving the tara, Peer always mentions the sack, barrel, wood, etc., to which it refers. But in some cases he does not use the word tara, following Widman in his mode of describing this deduction: *e.g.* on D 7 in a question relating to two barrels of soap he has "vñ geet ab an den zweyen lageln für den holtz 72 lb".

Peer is the first writer I have met with who explains the distinction between 'tara auf den centner' and 'tara in den centner'. This occurs in D 2′ and D 3 under the heading "Ein vnterricht, wie man den thara auff oder in cētner versteen sol". He says that the wood may be taken into account by giving something to the centner, or by deducting its weight from the centner*, and his examples show that if the tara is on the centner it is to be added to the centner, and if it is in the centner it is to be subtracted: thus if the tara is 10 lb 'auff den centner', 100 lb out of every 110 lb is 'lauter', and if it is 'in den centner', 90 lb out of 100 lb is 'lauter'.

§ 98. Peer gives (on F 4′) Widman's verses† (§ 89) about the Regula Fusti, followed by his example in which 27 ct 81 lb of cloves are bought, 13 lb in each centner being fusti, the pure cloves costing 11 ss 3 hlr a lb and the fusti 21 hlr (§§ 19, 65, 70). This example he works out in detail. He also gives another example in which 5 centners of cloves are taken from Venice to Nuremberg 'vnd gehet ab für den sack 6 lb', and 100 lb contain 15 lb fusti, and 1 lb 'lauter' is worth 13 ss and 1 lb of fusti is worth 2 ss.

Previously (on D 2′) he had given two questions in which saffron contained 'vnreins', which was not paid for. In the first, 100 lb of 'wilder saffran' cost 80 fl and the question is to find how much 7 ct 84 lb cost, if 100 lb 'haben 15 vnreins'. The next question is Widman's saffron question (§ 89), viz. if 100 lb of saffron cost $94\frac{1}{3}$ fl, what is the cost of $384\frac{1}{4}$ lb if '100 lb habē 15 lb vnreins'. It is to be remarked that although Widman places this question under Regula Fusti, Peer places his saffron question in which there is unrein (*i.e.* waste, of no value) in an earlier part of the book, and does not include unrein under fusti.

* "Ist die gemain regel, so etwas auf geben wirt zum centner sol man addirn, tregt ein kleyns mer, dadurch dz holtz bezalt wird. Rechent man aber das vnter einem cētner schwer etlich pfundt holtz sein sol, oder darfür lässt abgeen, sol man solche gemelte pfunt von einem centner subtrahirn. Des gleichē hab solche auffmerkung der rauchen war halben".

† They are given without a break (as in Widman), and not as verse (as in Böschensteyn): there are slight verbal differences.

Apianus (1527), § 99.

§ 99. Apianus's use of the word tara is similar to that of Rudolff and Peer, and, like them, he always mentions that the tara is for the sack, or barrel, or wood, etc.

In the oil question (quoted in § 71), which was modelled upon Widman's fig question (§ 11), he differs from Widman by using the word tara. Thus Widman wrote "Nu solt du fur holcz abschlahn̄ albeg fur eyn lagel 24.lb vn̄ dz ist 13 mol 24", and Apianus has "thara vor das holtz an jtlichem Vass abgeschlagen 19 lb . . . Multiplicir 9 lagel mit 19 lb thara".

Under the heading Regula Fusti (on M i) Apianus gives three examples relating to pepper, saffron, and cloves. In the first, 4 ct 38 lb of pepper is bought: 1 centner contains 7 lb of fusti and 1 centner of pure pepper costs 60 fl and 1 lb of fusti 9 hlr. In the second question (which has been partially quoted in § 71), '1 stumpff Saffran' weighs 244 lb 'thara für den stumpff abgeschlagen' $2\frac{1}{2}$ lb, 1 centner contains 20 lb fusti, and 1 centner 'lauter' costs 400 fl and 1 lb of fusti $1\frac{1}{2}$ fl. The third question (M i') relates to cloves, but it is unintelligible, sufficient data not being given, and the result is inconsistent with the data: but it is clearly meant that each centner contains a definite amount of fusti, and that the pure cloves and fusti have different values.

It will be noticed that in the first of these questions the refuse in the pepper is called fusti, and that it is paid for.

Rudolff's Exempel Büchlin (1530), § 100.

§ 100. Rudolff's *Exempel Büchlin*†, referred to in § 72, contains a number of examples in which tara (as a definite amount or 'auf den centner' or 'in den centner') and fusti occur. In one question (107)‡ he explains the difference between tara on the centner and in the centner. "Die rechnung darauff zu gründen, das durch in cen̄ 5 lb verstanden werden, 100 lb. geben lauter 95 lb. Vnnd durch auf den cen̄ 5 lb. das 105 lb geben 100 lb lauter"; and he points out that the word 'auff' is more favourable to the seller.

In some questions Rudolff uses the word 'Gerbeluer', which has much the same meaning as fusti. Thus question 105 is "Ein sack pfeffer wigt 5 centner, 18 lb. Tara für den

* This question remained unaltered in the second and third editions (Frankfort, 1537, 1541), except that in the third edition there is a misprint of 3 for 13.

† As in § 72 the quotations are from the edition of 1561, and the questions are referred to by their numbers.

‡ In this question, which relates to 'ein lagel seiffen', he refers to 'holtz oder thara gewicht'.

sack 4 lb $\frac{1}{2}$. Helt der cen̄. 13 lb Gerbelür vnnd werden 3 lb gemelts Gerbeluer pro 2 lauter im kauff angeschlagen, vnnd das lb lauter pro 3 ss $\frac{1}{2}$. Wieuil gelts kost der Pfeffer samentlich". In the working 13 lb Gerbeluer is taken as $8\frac{2}{3}$ lb pure: this added to 87 lb makes $95\frac{2}{3}$ lb, which at $3\frac{1}{2}$ ss the lb gives $334\frac{5}{6}$ ss. The result is then obtained as the fourth proportional to 100, $334\frac{5}{6}$ ss, $513\frac{1}{2}$.

The word Gerbeluer means garble, *i.e.* merchandise (generally spices) which contains an admixture of refuse or waste.

In the next question (106) some sacks of ginger weigh 16 centner 28 lb. The tara for the sack is 18 lb. "Helt der cen̄. 13 lb Gerbeluer vnnd 3 lb staub, werden angeschlagen im kauff 5 lb Gerbeluer pro 3 lb lauter". The centner pure costs $54\frac{1}{2}$ fl.

Here the 13 lb Gerbeluer is equal to $7\frac{4}{5}$ lb pure, and the dust is of no value, so that the centner contains $91\frac{4}{5}$ lb pure, which is paid for at $54\frac{1}{2}$ fl the centner, giving $50\frac{31}{1000}$ fl, and the first three terms of the proportion are therefore 100, $50\frac{31}{1000}$, 1610.

Albert (1541), § 101.

§ 101. There are many questions in 11° (Albert, 1541) which involve tara, either as an absolute quantity or dependent on the centner, and generally Albert mentions the receptacle to which the tara relates.

In a question (on I vi) relating to barrels of raisins there is 'Tara fur holtz 16 pfund', and in the solution he writes "Das Tara nim allweg vom hintern, Stehet aber das wörtlin, jnn, oder auff, dabey, so addirs dem fördern". On I viii he has a question relating to '4 Tonnen' of honey with 'Tara auff iglichen centner 15 pfund'. Here the centner is 110 lb so that 125 lb is given for the price of a centner and the explanatory words are "Tara auff, Addir (wie oben angezeigt) zum fördern". He does not seem to give any questions in which the tara is in the centner.

On I viii he has the heading "Die Regula Fusti, wird gemeiniglich gebraucht jnn den Neglein, Saffran, Corbern, Gold, Silber etc. wie hernach wird angezeigt", and he gives the example "Item, Ein sack mit Corbern wigt $8\frac{1}{2}$ cent. 34 pfund, Tara fur den sack 11 pfund, Kost 1 cent. lauter 3 fl 13 ss 4 hel. Vnd 1 cent. helt 13 pfund, Fusti, das ist vnreines. Facit 28 fl 3 ss 2 heller $\frac{14}{55}$ teil". Here the word fusti is used as equivalent

to vnreines or waste, and is not paid for. Corbern is presumably corbeeren, the fruit of the cornel tree.

In another question on Ki a person buys "einen sack mit Neglein, weget 3 ct — 16 lb, Tara fur den sack 7 pfund, Kost ein pfund Lauter 1 fl 13 ss*, Vnd ein pfund Fusti 9 heller, Vnd ein cent. helt 12 lb Fusti". Albert gives the solution in full, finding first the amount of fusti, then the cost of the pure cloves and the fusti, and adding the two amounts.

Spenlin (1546), § 102.

§ 102. In 14° (Spenlin, 1546), under the heading *Regula Fusti* (p. xlv.), there are a number of questions involving tara. It is explained that 'thara' is a deduction, as when one weighs wool in sacks, or fat, oil, or other substances in vessels, and then the sack or vessel alone, and so much is taken away from the whole amount. In his examples the tara is nearly always the weight of the sacks or vessels; but on p. xlvii. he gives a question relating to wool in which the 'tara fur ausswurff, auff 1. cent' is 10 lb, and the cost of the pure wool is $12\frac{1}{2}$ fl. Here the 'ausswurff' is the rejected matter. In connection with this question he mentions the distinction made by some calculators between 'auff' and 'inn', but states that he himself will always use 'inn'. He carries this out in his next question relating to the purchase of 2345 skins, in which "man gibt jm jnns 1000. fell 25. fell für ausswurff"†, viz. he finds a fourth proportional to 1000, 975, 2345, which is $2286\frac{3}{8}$, and this is the number of skins paid for. Here a deduction for waste is regarded as tara. In several other questions the tara is in the centner, and in no case is it on the centner.

It thus appears that Spenlin used the word tara not only for the weight of the receptacle but also for waste in the goods, and that both these deductions formed the subject of his chapter *Regula fusti*. Under this heading he gives no example in which fusti occurs, although there are two questions relating to cloves (in which there is tara for the sack or sacks), but on p. lxv′ (under 'Gewinn vnnd verlust') he gives a question in which a sack of cloves weighs 2 cent. $18\frac{1}{2}$ lb, "thara für den sack $7\frac{1}{4}$ lb, helt der cent $12\frac{1}{2}$ lb fusti oder still, kost 1. lb lauter 12. batzen 8.ϑ, vnd 1. lb fusti 7 batzen 1. cr . . .". This is an ordinary fusti question, and he explains fusti as still (stalks).

* Misprinted 1 fl 1 ss in the question.

† On p. lxix. he has another question of the same kind relating to wool, in which the 'thara für auswerffen' is 4 lb in the centner.

Riese (1550), § 103.

§ 103. In 15° (Riese, 1550) there are a number of questions in which tara is used, both as a fixed amount to be subtracted, and as dependent upon the centner. There is a special heading (pp. 25′ and 85) "Vom tara auff vnd in den Centner", in which the two methods of treating tara are illustrated by examples. None of the examples differ in principle from those given in earlier works. He has questions involving cloves and fusti at different prices, and in which the tara is given (pp. 31′, 32, 90).

Usual meanings of tara and fusti, §§ 104–105.

§ 104. It thus appears that the writers subsequent to Grammateus almost invariably used the word tara in connection with the weight of the vessel or other receptacle in which the goods were packed. When the tara is an absolute quantity, as for the sack or barrel, it is the actual weight of the receptacle, and when it is 'on' or 'in' the centner it is intended to represent this weight. The one exception I have met with is in Spenlin, where 'thara fur ausswurff' occurs, the 'thara' being the deduction made from the weight on account of the rejected matter in the merchandise itself.

Widman did not use the word tara. In his fig question he made a deduction for the weight of each barrel, his words being "fur holcz abschlahn̄ albeg fur eyn lagel 24lb"; but Apianus, in his oil question, which was closely imitated from it, replaced this by "thara vor das holtz an jtlichem Vass abgeschlagen" (§ 71). Widman in other questions also made a deduction for the weight of the receptacle. In Grammateus tara was used for waste, but the word only occurs once.

The distinction between auf and in den centner seems to have been made early, though the first explicit statement of it that I have found is in Peer (1527). It may be noted that the deductions for the weight of the vessel which Widman makes (other than those in which the actual weight is given) belong to the class 'in den centner'; but deductions 'auf den centner' are more usual in all the later Arithmetics. Böschensteyn uses 'am centner' with the meaning of 'auf den centner'.

§ 105. The word fusti, which occurs so frequently in connection with cloves, means the stalks, etc., mixed with them, which were paid for at a lower rate than the pure cloves. The word fusti was also used in connection with saffron and pepper (Apianus) and with corbern (Albert). In all the

questions I have met with in which fusti occurs in connection with cloves, it is paid for; but in Apianus's questions on pepper and saffron, and Albert's on corbern, it is regarded as waste. Widman and Peer used unrein instead of fusti when the matter was valueless, but Lacher has 'fusti vel impuri' and Albert 'Fusti, das ist unreines', showing that they regarded the words as having the same meaning.

Tropfke's remarks on tara, §§ 106–107.

§ 106. Under the heading 'Die Tararechnung' Tropfke has made some statements in his *Geschichte** which seem to be inaccurate. He writes: "Im *Bamberger Rechenbuch* von 1483 wird ein Gewichtsabzug als '*das Minus*' bezeichnet; die Behandlungsweise solcher Aufgaben im Rechenbuch (1489) des Johannes Widmann von Eger geht unter dem Namen *regula fusti*. Richtiger sagt Koebel 1537 *regula fusci* (fuscus = braun, unrein) und erklärt: 'Das wort Fusci, bedeut nichts anders dann ein zerbrochen gut gemülb, odder ander vnreynigkeyt, so in der Specerei funden wirt, als vnd den Negelin, Ingber, Saffran etc. Auch Silber vnderm golt, Kupffer vnderm sylber. Die vass vom Honig, Butter, Oley etc. vnd der gleichenn vermischt unreinigkeit' ... Dieser allgemeinen Erklärung gemäss behandelt Koebel auch Legierungsaufgaben in seiner regula fusci".

In the statement about the use of 'das Minus' he follows Cantor. This use of minus will be referred to in § 108.

Tropfke is incorrect in stating that Widman treated weight-deductions under *Regula Fusti*. Of the two examples which Widman gives under this heading one relates to fusti, which is paid for, and the other to 'vnreyn', which is not paid for; but there is no deduction for the weight of the wood, although such a deduction is made in other examples (§ 89).

Koebel was wrong in replacing fusti by fusci, and imagining that fusci was the correct form, and it is surprising that Tropfke should have accepted his supposed correction. Tropfke proceeds: "Das Wort Tara scheint in deutschen Rechenbüchern zuerst von Riese benutzt worden zu sein; sein Rechenbuch von 1550 weist es einmal in einer Überschrift auf: 'Von Tara auff und in den Centner', im Laufe der Rechnung erscheint aber immer das altgewohnte Wort Minus. Auch im Rechenbuch 'auff der linihen' von 1518 ist Tara im Text (3. Aufl., Erfurt 1530, Signatur C_{iiii}) gelegentlich verwendet".

* *Geschichte der elementar-mathematik* (1902), vol. i., pp. 112–113.

Tropfke's remark that in the course of the calculation the long-used word minus always appears might convey the impression that minus was used for tara in the working of the questions, but Riese's 'das minus' is distinct from tara, being the sum of the other quantities which, besides the tara, have to be subtracted. Thus he says "Nim ab das minus vnd tara" in questions 118, 122, 124, 125, etc. (pp. 24, 24′), and in 131 (p. 25′) he has "Nim ab das minus vnd nicht das tara, sonder gib das zum Centner"; and "nim ab das minus" occurs in 173 (p. 34), where there is no tara.

From the occurrence of tara in the 1530 edition of Riese's *Rechenung auff den Linihen*, Troptke seems to infer that it was used in the edition of 1518, of which Unger knew of no existing copy (page 31, note). It occurs in the 1525 edition of the *Rechenung*, first published in 1522 (§§67, 93), but had been previously used by Grammateus.

§ 107. Tropfke quoted from a book of Koebel's of 1537, but the word fusci and the same explanation of it also occur in Koebel's *Eyn neüw Rechēbüchlin of* 1525*. After the explanation which Tropfke has quoted (p. clxxxvi.) Koebel treats separately the three classes into which he has divided 'fusci' and he gives one example of each class. The three classes are (1) spices, (2) gold and silver, (3) oil, butter, honey, etc. In the example of the first class, a sack of cloves contains a certain proportion of fusti, the pure cloves and the fusti being paid for at different prices; and in the example of the third class, the weight of the containing vessel is calculated and subtracted from the gross weight, no account being taken of any impurities in the merchandise itself. The latter question is: the oil in three vessels weighs 370 lb; 3 lb in every 50 lb is taken off for the wood; and 1 lb of oil costs 2 ss; what is the value of the oil? In the solution the weight of the wood is found to be $22\frac{1}{5}$ lb, so that the question is merely

* Tropfke gives the title of the book from which he quotes as "Zwey Rechenbüchlein: uff den Linien und Zipher mit eynn angehenkten Visirbuch" (Oppenheim, 1537/38). The title of the book of 1525 is "Eyn neüw Rechēbüchlin Iacob köbels statschreibers zü Oppenheym auff den Liniē vñ spaciē gãtz leichtlich Rechen zü lernen mit vyelen züsetze, Nemlich der Regeln Fusci vff Specerei Goldt, Silber, Oley, Honig, etc. Darzü die regel Pagamēts eyn müntz gegen andere: züuergleichenn, etc." At the side there is a woodcut of a seated monkey, and below a 'Rechenbanck' with three 'Bankirs' and "Getruckt zü Oppenheym. . . . Anno M.D XXV." It will be noticed that 'Regula Fusci' occurs in the title. I do not know if this is a first edition or the first work of Koebel's in which 'Regula Fusci' appears. It does not occur in "Eyn Neüw Rechēbüchlein. Vff den Linien vñ Spaciē . . ." referred to in the last note on p. 29. In the second edition of Apianus's *Rechnung* (Frankfort, 1537) fusti is replaced by fusci.

to find the cost of 347$\frac{4}{5}$ lb at 2 ss a lb, which is 34 gn 15 ss 7$\frac{1}{5}$ hlr*.

The question about cloves is: a sack of 336 lb of cloves contains pure cloves and fusti: 1 lb of pure cloves is worth 9 alb, and 1 lb of fusti is worth 11 pence; and it is found that 5 lb of cloves contain 8 lot of fusti: it is required to find the weights of pure cloves and of fusti, and the value of the whole. Here $\frac{1}{20}$ of 336 lb is fusti and the remainder is pure cloves, *i.e.* there are 16$\frac{1}{2}$ lb 9$\frac{3}{5}$ lot of fusti and 319 lb 6$\frac{2}{5}$ lot of pure cloves. [Koebel gives the fractions as $\frac{3}{8}$ and $\frac{5}{8}$ instead of $\frac{3}{5}$ and $\frac{2}{5}$.] The former at 11 pence a lb amounts to 23 alb 0$\frac{4}{5}$ d, and the latter at 9 alb a lb, to 191 lb 7 alb 6$\frac{2}{5}$d, taking 8 pence to the alb and 15 alb to the lb. Thus the total value is 193 lb 0 alb 7$\frac{1}{5}$ d. Koebel's values are 23 alb 4$\frac{3}{4}$ d, 191 lb 7 alb 4$\frac{3}{8}$ d, and 193 lb 1 alb 1 d for the total†.

Koebel's division into classes is unsatisfactory, for the spices need something to contain them (such as a sack) as much as the oil, etc., need a barrel or jar; and the latter, as Koebel carefully points out‡, contain impurities (either to be paid for at a lower price, or not to be paid for) as well as

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* p. cxcviii. Koebel erroneously gives the result as xvii gn vii ss ix hlr $\frac{\text{iii}}{\text{v}}$, which is only half the correct amount. In this book of 1525 (as in his earlier writings) Koebel, although he mentions the Arabic figures, uses cumbrous Roman numerals even in fractions. Thus in this question he gives the weight of the oil as CCC L XX lb, and takes off III lb in L lb for the wood, the cost of I lb of the oil being II ss. From the proportion L III III$^{\text{C}}$ LXX he obtains 'XXII pfūd $\frac{\text{i}}{\text{v}}$ eines pfūds' as the amount taken off for the wood, and he then forms the proportion I II CCCXLVII $\frac{\text{iiii}}{\text{v}}$ to give the final result. We may presume that Koebel supposed he was making a correction in writing fusci instead of fusti, but he was behind his age in not using Arabic figures, and his books afford striking evidence of the awkwardness of the Roman numerals in expressing operations and results. On the other hand he was in advance of his time in numbering the pages (instead of leaves), for, after ix, where the arithmetic begins, the pages are numbered (in Roman numerals).

† Koebel obtained his values by reducing the weights to lots and omitting the fraction of a lot; thus his proportions are 32 : 11 :: 537 and 32 : 9 :: 10214; but these proportions give the values 23 alb 0$\frac{19}{32}$d and 191 lb 7 alb 5$\frac{1}{2}$d, which differ in the pence from Koebel's results. [It is possible that he may have taken the former value to be 23$\frac{19}{32}$ alb, which would give 23 alb 4$\frac{3}{4}$d]. The proportions and results are expressed in the Roman numerals. In the Frankfort edition of 1544 (and perhaps in earlier editions) the Roman numerals are replaced by Arabic figures, but the leaves (not pages) are numbered. The questions that have been quoted occur on pp. 80, 80′ of this edition. The errors are not corrected.

‡ "Die vass vom honig, butter, oley etc. vn der gleichē vermischt unreynikeyt, das von dem gantzen vnd reynen in kauffen vñ verkauffen abgesundert, oder vm̃ minner gelt dañ das reyn vnd güt kaufft vnd verkaufft wirt" (p. clxxxvi); and the third class is introduced in the words, "Der drit vnderscheyt der regel fusci leret rechnen vnd enscheyden den kauff des öles, buttern, honig, vnd der gleichen geware, die in vassen verkaufft vñ kaufft wirt durch regeln vnd exempeln wie nachuolgt (p. cxcvii): and then follows the oil question. These quotations are from the 1525 edition.

the spices. In his example relating to cloves there is no mention of the weight of the receptacle in which they are packed, and in the oil question no impurity or waste is taken into account. The word tara is not used by Koebel.

The use of 'das minus' in the Bamberg Arithmetic (1483), § 108.

§ 108. Cantor states that in Chapter X. of the Bamberg Arithmetic "what, on account of the packing, was not to be included in the weight of the goods, and which was later called tara, is here simply called *das minus* and is subtracted".* Without examining the book itself it is not possible to judge whether 'das minus' was used as a special term for tara, or merely meant that it was a minus quantity and therefore to be subtracted. The latter view would seem the more probable. In his fig question Widman defines − as minus and gives the direction 'add the −, that is 75 lb' (addir dz − dz ist 75 lb). Here 'dz −' is equivalent to 'das minus', and is merely the sum of the negative weights, which is to be added to the weight of the wood (equivalent to tara): and in another question (§ 26) he gives the direction 'add + and − together', meaning that the terms which have these signs are to be added (as they are on opposite sides of the equation). Riese (1550), as has been mentioned, often has 'Nim ab das minus vnd tara' (§ 106).

The use of the words tara and fusti by Italian writers, §§ 109–112.

§ 109. The early Italian writers used tara in a more general sense, for it was applied to any deduction from the gross weight of merchandise that had not to be paid for. It was almost invariably expressed as a percentage.

Borgi (1484), §§ 110–111.

§ 110. In Borgi (1484)† there are many questions in which 'abatando de tara' occurs, the tara being almost always expressed as a percentage on the gross weight. A typical question is: 100 lb of cotton cost 6 duc. 7 gr. 18 p., what will 5432 lb cost, 'abatando de tara' 7 lb per cento, and the 'mesetaria' 2 duc. per cento?‡ The procedure is: calculate the amount of the tara, and subtract it from the gross weight,

* Cantor ii. p. 224. "Was wegen Verpackung nicht als Waarengewicht mitzurechnen ist und später Tara genannt wurde, heisst hier einfach *das Minus* und wird subtrahirt".

† The title of Borgi's book was given on p. 4.

‡ p. 46′. Mesetaria was a percentage on the total sum paid, which was due to certain officials.

giving the net weight (nete de tara): then calculate the cost at the price given, and deduct 2 per cent on this cost for mesetaria. Borgi gives a great many similar questions* in which the tara is always a percentage on the weight.

In one question† a mier of oil is worth 34 duc. 17 gr., and the question is to find the value of 17 miera 16 miri 19 lb 'abatando de tara' 3 miri 15 lb per mier, and the 'mesetaria' $2\frac{5}{6}$ per cento. A mier is 1000 lb and a miro 25 lb, so that 3 miri 15 lb is 90 lb and the tara is 90 lb on 1000 lb, that is 9 per cent; thus the tara is expressed practically as a percentage.

In another example the equivalent to 'tara auf den centner' occurs, though the word tara is not used. It relates to skins: if 100 (vn centenar de pele) are worth 13 duc. 10 gr., what will 1312 be worth, 8 per cent extra being given (dagandone sopra pele. 8. per cento)? Thus 108 skins are given for 13 duc. 10 gr.‡

§ 111. Borgi has two questions relating to cloves and fusti. In both questions a definite portion of the fusti is not paid for, and the rest is paid for as if it were pure cloves. The portion not paid for is called the tara.

In the first question 1 lb of cloves which holds 18 sazi of fusti is worth 7 grossi, what is the value of 594 lb?§ It is explained that by custom 2 sazi are subtracted from the amount of fusti in the lb, and that half of the remainder is the tara, the rest being paid for as if it were all pure cloves. Thus in this case 2 sazi are subtracted from 18 sazi, leaving 16 sazi, and half of this, viz. 8 sazi, multiplied by 594 gives 66 lb, which is the tara on 594 lb. Subtracting 66 lb (che e la tara de 594 lb) there remains 528 lb, which is 'el neto de tara', and at 7 gr. the lb is worth 154 ducats.

In the second question‖, which is of the same kind, and subject to the same rule, 1 lb of cloves, which holds 15 sazi 6 karats of fusti, is worth $8\frac{1}{3}$ grossi, what is the value of 490 lb 7 oz? Here, taking off the 2 sazi, 13 sazi 6 karats are left: the half of this is 6 sazi 15 karats which multiplied by $490\frac{7}{12}$

* pp. 47–51′.

† p. 55′.

‡ p. 63.

§ p. 62. The meaning is that 1 lb of cloves holds 18 sazi of fusti, and that 1 lb of *pure* cloves is worth 7 grossi. A similar want of exactness occurs in many other questions, but the meaning is generally free from ambiguity. The tables of weight and money used by Borgi are: 24 karats make a sazo (or sazzio or saggio), 6 sazi (or sazzi or saggi) make an ounce, 12 ounces make a pound: 32 piccoli make a grosso, and 24 grossi make a ducat.

‖ p. 62′.

gives 45 lb 1 ounce 4 sazi $2\frac{3}{4}$ karats, which is the tara on 490 lb 7 oz (e tanto tien de tara 490 lb 7 oz).

It will be seen that there is no separate price for the fusti: its inferiority to the cloves is taken into account by the fact that a portion of it is not paid for.*

Calandri (1491), *Pellos* (1492), § 112.

§ 112. In Calandri (Florence, 1491) there are no questions involving tara, nor is the word used. In Pellos (Turin, 1492) the word seems only to occur on (p. 59) in connection with silver which besides fine silver contains 'dross or tara' (brut ho tara).

Paciolo (1494), §§ 113–114.

§ 113. In Paciolo† (1494) there are several questions of the same kind as those in Borgi, in which tara is given as a percentage, and in one question (relating to wool) the tara is 8 lb in 1000 lb. The word datio is generally used instead of messetaria for the percentage tax on the total cost of the merchandise.

In the first example 4 per cent is taken off for tara (abattēdo tara . 4.lb per cēto)‡; in the next $6\frac{1}{2}$ per cent is taken off for dono (abattendo di dono . $6\frac{1}{2}$. lb. per cēto), and in the next 3 per cent is taken off for usage (abattendo per vsāza . 3 . lb per cēto). The deduction for usage is treated as tara, but dono is equivalent to 'tara auf den centner', *i.e.* in this example $6\frac{1}{2}$ is given with the 100, so that the gross weight is reduced in the proportion of $106\frac{1}{2}$ to 100 to obtain the net weight. Thus tara is a percentage to be subtracted, and dono is an extra amount which is given with 100 lb§.

In another example in which 100 lb of new wax is worth 12 ducats and 100 lb of old wax is worth 8 ducats, it is required to find the value of 987 lb of wax of which 46 per

* If 1 lb of cloves contains l sazi of fusti, this rule is equivalent to paying for all the fusti at the rate of $\left(\frac{1}{2} + \frac{1}{l}\right) g$ per lb, where g is the value of a pound of pure cloves.

† "Sūma de Arithmetica Geometria Proportioni & Proportionalita" (Venice, 1494. See § 6 (p. 4).

‡ In this example the amount of the merchandise is 987 lb, on which 4 per cent is $39\frac{48}{100}$, and Paciolo says that by mercantile usage the fraction is to be ignored, and 39 taken as the tara because $\frac{48}{100}$ is less than $\frac{50}{100}$; but if the remainder had exceeded half of the divisor the tara would be taken to be 40. Borgi did not take the nearest whole number: he merely discarded the fraction. Thus in a question on p. 49′, in which tara of $8\frac{2}{3}$ per cent is to be taken from 3817 lb, he finds that this amounts to 330 lb and $80\frac{2}{3}$ hundredths of a lb, but he throws off the fraction and treats it as 330 lb. It will be seen that succeeding writers followed Paciolo's rule.

§ When the abatement is r per cent for tara, the gross weight is reduced in the proportion of 100 to $100 - r$, and when it is r per cent for dono, it is reduced in the proportion of $100 + r$ to 100.

cent is old, taking off dono at 3 per cent from the new, tara at 4 per cent from the old, and datio at $3\frac{1}{2}$ per cent. This example is clearly intended to exhibit the difference between tara and dono.

In another question the value of 9876 lb of red copper is required, each 1000 lb containing 250 lb of tin, 643 lb of copper, and the rest of lead, the prices of tin, copper, and lead being 90, 96, and 24 ducats respectively per 1000 lb, 4 per cent being taken off from the tin for dono, 10 lb per 1000 lb from the copper for tara, and 12 lb per 1000 lb from the lead for waste (abattendo dono del stagno . 4.lb per cento: e tara del ramo . 10 lb per migliaro: e callo del piombo . 12.lb per migliaro). There is also a final charge of 6 per cent for the tax and expenses. In this example three different kinds of deduction, dono, tara, callo occur.*

§ 114. Paciolo gives an example involving pure cloves and fusti in which the fusti is treated in the same way as in Borgi's two examples. In this example 1 lb of cloves is worth $5\frac{1}{4}$ grossi, and holds 12 saggi 20 karats of fusti and leaves (antofani), and it is required to find the value of 2400 lb. It is explained, as in Borgi, that 2 saggi are taken away by custom and half of the remainder is the tara, which in this case is 5 saggi 10 k. This multiplied by 2400 gives 180 lb 6 ounces 4 saggi, which subtracted from 2400 lb leaves 2219 lb 5 ounces 2 saggi; and this at $5\frac{1}{4}$ grossi the lb produces 485 ducats 12 grossi $2\frac{2}{3}$ piccoli†.

Paciolo also calculates the messetaria at 2 per cent on this amount.

There is also a question in which the fusti is paid for separately, viz. 1 lb of pure cloves is worth 6 gr: 1 lb of fusti is worth 3 gr: and 1 lb of husks (capelletti) is worth 2 gr: what is the value of 2400 lb which contain 12 saggi of fusti per lb and 14 saggi of husks per lb, taking away 2 (or 3) per cent for messetaria?

Tagliente (1515), § 115.

§ 115. Tagliente in his *Libro de abaco*‡ (1515) has questions and explanations similar to those in Borgi and Paciolo, some

* The examples referred to in the text are nos. 11–16 on pp. 61, 61′.

† p. 62 There is a mistake in Paciolo's working of this example: he gives 485 duc. 11 gr. $6\frac{2}{3}$ as the value. The mistake occurs in the reduction of 2219 lb 5 ounces 2 k. to karats.

‡ The title of the book is "Libro de abaco che insegnia a fare ogni raxone marcadantile & apertegare le terre con larte di la giometria & altre nobilissime raxone straordinarie cō la tarifa come raspondeno li pexi & monete de molte terre del mondo con la inclita citta de. Venetia El qual Libro se chiama Texauro vniuersale Concesso per lo Serenissimo Dominio Venetiano per anni diexe cū grā".

of the questions being identical. His explanations of the terms involved are more detailed: thus in the explanations of tara he states that it is a deduction arranged between the buyer and seller on account of impurities in the merchandise, or for any other cause, and he mentions that in deducting the tara it is usual to take its value to the nearest lb, thus neglecting a fraction less than $\frac{1}{2}$ lb. This explanation he attaches to the first question in which tara occurs (no. 54, F iiii'), viz. if 100 lb is worth 4 ducats what will 987 lb be worth, taking off tara at 4 per cent? This is also Paciolo's first question in which tara occurs (except that Tagliente has changed 16 per cent to 4 per cent); and it is the question in which Paciolo explains that the tara is to be taken to be the

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In the introductory address Hieronymus Tagliente (Hieronimo taiēte Citadin venetiano), after praising arithmetic "which is called one of the seven liberal arts", states that in his youth, with the help of his kinsman and master Giovanni Antonio Tagliente, pensioner of the Venetian State, he diligently studied the works of the best authors and was led to compile the present work (Di che in questa mia verde e iuuenil etade ho voluto cū laiutto del mio clarissimo cōsanguineo & preceptore Mēser Iouāni Antonio Taiente pronisionato per sne virtu dal Serenissimo dn̄io Venetiano. vedere con ogni studio & diligentia diuerse oppere fabricate per excelētissimi auctori. & nō cō poca mia fatica & industria ho voluto cōmulare & cōponere la pressēte oppera). A second edition of this work was published in 1520. In 1525 Hieronymus Tagliente published an enlarged edition, almost amounting to a new work, which forms the subject of § 116.

In 1527 the elder Tagliente published an Arithmetic with the title "Opera nova che insegna a fare ogni ragione di mercantia. Et prima a sapper releuare ogni numero, Poi a Moltiplicare, Partire, Somare, Sottrare con le sue proue, la Regola del tre con laquale si puo fare ogni ragione di Mercātia, cioe come saria a dire, Se la libra, el centenaro o uer el migliaro di una Mercantia uale tanti danari, che ualera tante libre. Et anchora a sapper fare le Ragioni delle Compagnie & baratti con altre molte ragione. Et a pertegare le terre & muri con arte Giometricale come nell-opera uederete". The colophon is "Stāpado per Bernadin Venetian de Vidali, M, D, XXVII". I have given in full the title of this book (which is in my own possession) because I have not met with any description of it, or found any reference to it indicating that it is different from that of 1515. It is put forth as the work of Giovanni Antonio Tagliente; but in the book of 1515 he is mentioned merely as having assisted Hieronymus Tagliente in the preparation of that work. In the address to the readers (Alli benigni lettori. Iovani Antonio Tagliente) nearly the same words occur, as were used by Hieronymus in 1515, in stating that having studied the best authors (hauendo io veduto Diuerse opere fabricate per Excellentissimi Auttori, & nō con poca mia fatica) he has wished to compose the present little work which "will teach and instruct lucidly with great facility and brevity". As regards the contents of the book, a great portion of it is almost a reprint of that of 1515, but there are some variations. The paragraph numbered 20 in the 1527 edition does not appear in that of 1515, so that although those numbered 8 to 12 and 14 to 19 are the same in both editions, those numbered 20 to 137 in the 1515 edition correspond very nearly to 21 to 138 in the 1527 edition, 138 being the last of the numbered paragraphs in this edition. The 1527 edition is thus similar to that of 1515 in contents, and it has the same woodcuts in the text, but it is different in form, being of quarto size, while the editions of 1515 and 1525 are octavo, as also are all the reprints. There is also a different engraving on the back of the title-page.

Besides the editions of 1515, 1520, and 1527, and the book of 1525, which all bear the name of Tagliente, there are a great number of editions of the book of 1515 (but with different wood engravings) which have no author's name, date or place. These presumably are pirated editions. After a time they have date and place, but

nearest integral number of lbs. After no. 56 (G ii) Tagliente states that the messetaria in Venice, called gabella in other cities, is a tax or toll of 2 per cent (or more or less as notified) of the value of the goods, half of which is paid by the seller and half by the purchaser, and that the whole amount is handed over by the buyer to the tax-officer, so that if the amount due to the vendor is 100 ducats, the purchaser pays 99 ducats to him and 2 ducats to the tax-officer. In the examples, the percentage by which the messetaria is expressed refers to that portion of it which is paid by the vendor and not to the whole tax. Thus in no. 56 (G i′) where 100 lb of sugar is worth 15 duc. 14 gr. and the question is "che val lire. 9745. abbatendo de messetaria ouer de gabella ducati. 2. per .100.", the value of the sugar is found to be 1518 duc. 14 gr. 9 p., and 2 per cent of this, viz. 30 duc. 8 gr. 29 p., is subtracted leaving 1488 duc. 5 gr. 12 p.*, which is what the vendor receives, and presumably the sugar costs the purchaser 1548 duc. 23 gr. 6 p., of which he pays 60 duc. 17 gr. 16 p. to the tax-officer, who therefore receives 4 per cent on the value of the sugar.

After several questions (nos. 57, 65, 67, 68) involving both tara (always expressed as a percentage or with reference to 1000 lb) and messetaria, Tagliente gives (no. 81, I ii′) Paciolo's question about the 9876 lb of red copper, which was quoted in § 113. The numbers are the same, but the 4 per cent dono on the tin is replaced by a deduction of 4 per cent (abatēdo del stagno lire. 4. per cento & tarra del rame lire. 10. per miaro & per calo del piombo lire. 12. per miaro), the reason probably being that he has not yet explained dono.

Tagliente's name is still omitted. In vol. xvi of the *Atti dell' Accad. Pontif. de' nuovi Lincei* (1863), pp. 139–228, Prince Boncompagni has given an elaborate account of a great number of editions, and has reprinted from each a portion (viz. paragraph 8), occupying more than a page, for comparison with the original edition of 1515. Besides the numerous undated editions, Boncompagni describes others printed at Milan in 1541 and 1547 by Io. Antonio da Borgo, and at Venice in 1548 by Giouanni Padouano.

Prince d'Essling, in vol. iii., pp. 299–305 of "Les livres à figures Vénitiens de la fin du xv^e Siècle et du commencement du xvi^e" (Florence and Paris, 1909) gives an account of the books of 1515, 1520, 1525, and of eight other editions, and reproduces some of the engravings. He regards the earliest anonymous editions as pirated (contrefaçons), and, from their engravings and ornamentation, places the date about 1520. These pirated editions have on the border of the last page "Opus lucha ātoniode uberti fe ī uīnetia". Essling states that Lucas Antonio di Uberti was the engraver: but he has been supposed by some (*e.g.* by Libri) to be the author, and the book has been catalogued under this name. In *Rara Arithmetica* the work of 1525 is described, but the other editions are not differentiated from it. All the pirated editions that I have seen, or seen described, are copies of that of 1515. It may be mentioned that the misprints in them are more numerous than in the original edition.

* Through a wrong subtraction Tagliente gives this result as 1477 duc. 5 gr. 12 p. The error is repeated in the book of 1527 (no. 57, F iv′).

This explanation is given subsequently on Iiiii′ in the solution of no. 86 (which is Paciolo's question about the 987 lb* of wax quoted in § 113), where Tagliente states that the dono of 3lb per 100lb means that 103lb is given for the price of 100lb.

The only question involving fusti which he gives is the same as Borgi's question (quoted in § 111) of the 594 lb of cloves in which 1 lb of cloves contains 18 sazi of fusti: and he explains, as in Borgi, the procedure of taking 2 sazi from the 18 sazi and halving the remainder to obtain the tara.

Tagliente (1525), § 116.

§ 116. This work† of Hieronymus Tagliente differs from that of 1515 (considered in the preceding section) in being more comprehensive (*e.g.* it includes fractions) and in containing more examples: but the general explanations of tara, messetaria, etc. are omitted. There are a number of questions in which tara alone is deducted (K 4, K 4′, L 1, L 1′), in which messetaria alone is deducted (L 2, L 3′, L 4′), and in which both are deducted (L 2′, M 2). In the question on L 2′, the value of 675 lb [misprinted 375 lb] is required 'abbattendo de tara £ $3\frac{1}{2}$. e de mesetaria. $1\frac{1}{2}$. p cẽto'. He has one dono question‡ (M 1) in which 20 per cent is given, so that the purchaser receives 120 lb for the price of 100 lb.

There are two fusti questions, in one (O 3) of which 1 lb of cloves contains 18 sazi of fusti, and it is explained in the question itself that after deducting 2 sazi, half the remainder is tara (La £ de garoffoli netti val gr. 15 . . . che atien de fusti sazi . 18. p £ e sapi chel se da sazi . 2. per £ e del resto la $\frac{1}{2}$. son la tara). In the other (O 3′) the pure cloves and the fusti are paid for at different rates. A question of the same class (M 2′) relates to oil in which a mier of oil contains 13 miri 15 lb of pure oil, and the prices per mier of the pure oil and of the crude oil are given. There is also a question (O i) of the same kind as that relating to red copper in the earlier book of 1515 (§ 115), but without any tara on the lead.§

* Printed 387 lb, which is probably a misprint, as all the other numbers in the question are the same as in Paciolo: but the change is of no importance, as the answer to the question is not given.

† "Opera che insegna A fare ogni Ragione de Mercãtia Et apertegare le Terre Con arte giometrical Intitolata Componimẽto di arithmetica. Con gratiª & preuilegio M.D.XXV." In the address to the reader (Al benigno lettore Hieronymo Tagliente) the author refers to the earlier work (of 1515), in which he was helped by his kinsman, and says that he has been urged by his pupils and friends to compose a larger and more perfect work.

‡ "El 100 della chassia che se ne dona. 20. per cẽto val duchati. 18. gr. 20. che valera lb 1450. e batti de messetaria duc. 2. gr. 14. per cento".

§ There must be misprints in the question as the amount of copper and tin in 1 mier exceeds 1 mier.

Up to no. 50 (l iii′) the contents of this book (excepting for the portion relating to fractions) are generally the same as in that of 1515, but after no. 50 the questions and paragraphs (which are not numbered) are quite different up to X iii′, where a paragraph is numbered 143, and the paragraphs 143–158 (geometry) are the same as in the book of 1515.

Thus Hieronymus Tagliente has practically rewritten the arithmetic after no. 50.

It is strange that the work of 1515 which, like that of 1525, bears the name of Hieronymus Tagliente should have been republished (as mentioned in the note to § 115) in 1527 with the contents practically unaltered, but in a different form and with a different title, by Giovanni Antonio Tagliente, who was only mentioned in the edition of 1515 as having assisted Hieronymus in its preparation. So far as I know there was no other edition of the work of 1525 (which Hieronymus claims as entirely his own), nor was it pirated.

Feliciano (1526), § 117.

§ 117. Feliciano* only gives two questions involving tara: in one a tara of 6 per cent is deducted (abatendo de tara lire .6. per cento) and in the other there is a tara of 4 lb per cent and a messetaria of 1 ducat per cent (battendo de tara lire .4. per cento e pagar de mesetaria ducati vno per cento).

Sfortunati (1534), § 118.

§ 118. Sfortunati† gives Paciolo's two questions: if 100 lb is worth 16 lire what is the value of 987 lb, taking off 4 per cent tara; and if 100 lb is worth 12 lire what is the value of 987 lb, taking off $6\frac{1}{2}$ lb per cent for dono? In connection with the first he explains that the nearest integer (expressed in lbs) is to be taken as the tara. In the second question he explains that dono is the contrary to tara, as instead of being subtracted from the 100 lb it is added, so that the question becomes: if $106\frac{1}{2}$ lb is worth 12 lire, what is the value of 987 lb?‡

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* "Libro di Arithmetica & Geometria speculatiua & praticale: Composto per maestro Francesco feliciano da Lazisio Veronese Intitulato Scala gramaldelli:" (Venice, 1526). A reproduction of the title-page is given on p. 147 of *Rara Arithmetica*. The book also contains algebra, and there is not very much commercial arithmetic. The two examples involving tara occur on pp. G 4′ and H i.

† "Nvovo lvme libro di arithmetica . . . Composto per lo acutissimo prescrutatore delle Archimediane & Euclidiane dottrine Giovanni Sfortvnati da Siena" (Venice, 1534). A facsimile of the title-page is given in *Rara Arithmetica*, p. 176; and on pp. 174, 177 other editions are described.

‡ The questions described in the text are Propositions 33–39, pp. 45–46′.

The next question is: 100 lb of French wool is worth 16 lire 10 soldi, what is the value of 24 bales which weigh in all 840 lb, taking off for bands, cordage, and sacks $7\frac{1}{2}$ lb per bale, and tara 5 lb per 100? Here $7\frac{1}{2}$ multiplied by 24 gives 180 lb, which subtracted from 840 lb leaves 660 lb: the tara is 5 per cent of this amount, viz. 33 lb, and therefore the result is obtained by finding the value of 627 lb at 16 lire 10 soldi for 100 lb. It will be noticed that the deduction for the weight of the sacks, etc. is not called tara, and that the tara is a percentage on the actual weight of the merchandise itself.

In the next question he makes a deduction by usage (abbattendo per vsanza) of 3 lb per 100 lb, and of $1\frac{1}{2}$ per cent for datio. The deduction by usage is treated in the same manner as tara.

The next question relates to a quantity of new and old wax, the old wax, which is 45 per cent of the whole, being reduced by a tara of $2\frac{1}{2}$ per cent, and the new by a dono of 2 per cent, the datio on the whole being 3 per cent. He then has a question relating to three kinds of wool in which the deductions are respectively for tara, dono, and usanza, as well as the messataria.

This is followed by a question involving fusti in which 1 lb of cloves is worth $6\frac{1}{2}$ grossi and contains 10 saggi and 12 carats per lb of 'fusti and antofani'. He gives the usual rule of subtracting 2 saggi from the 10 saggi 12 carats and taking half the remainder for the tara.

Cataneo (1546), § 119.

§ 119. Cataneo* has a simple question in which the tara is 5 per cent, followed by one in which 100 lb cost 18 lire 10 soldi, and it is required to find the value of 3 bales weighing 750 lb, taking away 4 lb per bale for sacks and bands, and $5\frac{1}{2}$ lb per 100 for tara. Here the deduction for sacks, etc. is not called tara, and the tara is a percentage on the merchandise itself. In his next question a deduction of 3 lb per cent is made 'per usanza', and there is a datio of $2\frac{1}{2}$ ducats per cent.

He then gives a dono question, but uses the word dando: "che uarranno £.658. dando ne sopra £.5. per. 100?" He explains, as in Sfortunati, that this is the contrary to tara, and that the 5 is to be added to the 100.

* "Le pratiche delle dve prime mathematiche di Pietro de Catani da Siena libro d'albaco e geometria" (Venice, 1546). The title-page is reproduced in *Rara Arithmetica*, p. 213.

The next question relates to two kinds of wool, the tara on one kind being 4 per cent and on the other 3 per cent, the messetaria being 2 ducats per cent.*

He has no question in which fusti occurs.

Ghaligai (1548), § 120.

§ 120. Ghaligai† has only one question in which tara occurs: 'El Migliaio d' alcuna cosa' is worth 164 fl. 18 s. 3 d, what will be the value of 5876 lb 9 ounces, the tara being 5 lb 'per centinaio'?

Tartaglia (1556), §§ 121–123.

§ 121. Tartaglia's *Trattato di numeri et misure*‡ is a large and comprehensive work containing elaborate explanations of tara, messetaria, fusti, etc. He states that tara is a deduction from the merchandise of so much per lb, or 100 lb, or 1000 lb, or other definite weight or measure, on account of its being dirty or moist, in accordance with custom: and that messetaria is a tax in Venice which is paid both by the buyer and seller, and is a percentage on the amount paid for the goods and not on the goods themselves. The buyer retains the seller's portion of the tax, and pays the whole tax (*i.e.* his own portion and the seller's also) to the tax-officer. As an example, Tartaglia supposes that the messetaria is 2 ducats per cent, and that the amount is 300 ducats: the buyer then pays 294 ducats to the seller, and 12 ducats to the tax-officer.§ This explanation makes clear, what could only be inferred from Tagliente, that when the messetaria is stated to be, say, 2 per cent, this means that both the vendor and purchaser pay 2 per cent, so that the tax-officer receives 4 per cent.

Tartaglia then gives an example involving tara, and two involving tara and messataria, the tara being a percentage on the weight of the merchandise, and the messetaria a

* These questions occur on pp. 27, 27′.

† "Pratica d'arithmetica, di Francesco Ghaligai . . . " (Florence, 1548). The classes of commercial questions which involve a deduction for tara seem almost peculiar to the Venetian Arithmetics. In this work, published at Florence, there is but one such question, and in Calandri (Florence, 1491) and Pellos (Turin, 1492) there are none (§ 112). Ghaligai's book also contains algebra. The example quoted in the text occurs on p. 29′. This is the second edition, the original edition having been published in 1521. Ghaligai died in 1536 (Boncompagni, *Bullettino*, vol. vii, p. 485).

‡ "La prima parte del general trattato di nvmeri, et misvre di Nicolo Tartaglia . . . " (Venice, 1556).

§ This account of tara and messetaria occurs on p. 66′; but he gives on p. 151′ an even fuller explanation of messetaria, and states that the buyer has to keep back the seller's portion or else pay it himself.

percentage on the amount of money which is the net value of the merchandise.

He then gives a question relating to tin, in which the tara is 5 lb 8 oz per mearo [*i.e.* 1000 lb] and the messetaria is 4 duc. 18 gr. per cent. The next question relates to oil in which there is a deduction for waste (abbattendo di callo) of 6 lb 9 oz per mearo, and also a messetaria.

The next question relates to the value of 19 carghi 48 lb of pepper, 'abbattendo di tarra £9 oncie 2. per cargo', and also a messetaria and another tax.* A cargo is 400 lb. There is a similar question on p. 105 in which the tara is also 9 lb 2 oz per cargo.

§ 122. On p. 70 Tartaglia gives a question about cloves in which 1 lb of pure cloves is worth 16 gr. 11 p., and 1 lb or fusti is worth 3 gr. 8 p., and 1 lb of cloves contains 6 sazzi 7 carats of fusti. He states that in Venice the custom is for 3 sazzi of fusti in the lb to be paid for as pure cloves, and that only the amount of fusti in excess of 3 sazzi is to be paid for as fusti. Thus in this case 68 sazzi 17 carats is paid for as pure cloves and 3 sazzi 7 carats as fusti.†

But on p. 155′ he gives a question in which 3 sazzi of fusti in the lb is to be paid for as good cloves, and half of the remaining fusti is also to be paid for as good cloves, which he says is in accordance with custom, thereby following the rule given by Borgi and his successors, except that the amount of fusti paid for as pure cloves is 3 instead of 2 sazzi: but in the next question the values of the pure cloves and fusti are given, and 3 sazzi of fusti is treated as good cloves, as in the example just quoted from p. 70. Thus it is clear that in Tartaglia's time both customs existed.

* The explanation and examples mentioned in the text occur on pp. 66′–70. The examples are nos. 1–7.

† Tartaglia considers this procedure unfair, for in his view the rule that 3 sazzi of fusti in 1 lb of cloves should be treated as pure cloves means that for every 69 sazzi of pure cloves 3 sazzi of fusti should be counted as pure cloves, so that the amount of fusti to be paid for as pure cloves should be $\frac{1}{23}$ of the amount of pure cloves instead of $\frac{1}{24}$ of the whole amount of the pure cloves and fusti. It is clear that if the cloves contain a good deal of fusti the customary procedure is to the detriment of the buyer, and Tartaglia says the procedure was adopted to simplify the calculation and to benefit the seller.

If g is the cost of 1 lb of pure cloves and f the cost of 1 lb of fusti, and if 1 lb of cloves contains n sazzi of fusti, then (since there are 72 sazzi in a lb) the value of 1 lb of the cloves, according to the customary procedure, is

$$\left(1-\frac{n-3}{72}\right)g+\frac{n-3}{72}f, \text{ that is } \frac{75-n}{72}g+\frac{n-3}{72}f,$$

and according to Tartaglia it should be

$$\left(1-\frac{n}{72}\right)\left(1+\frac{3}{69}\right)g+\left\{1-\left(1-\frac{n}{72}\right)\left(1+\frac{3}{69}\right)\right\}f, \text{ that is } \frac{72-n}{69}g+\frac{n-3}{69}f,$$

the difference in favour of the seller being $\frac{1}{1656}(n-3)(g+f)$.

§ 123. On pp. 100′–106 Tartaglia gives a series of questions to explain the manner in which tara and messetaria are dealt with in Venetian practice. On p. 103 he states that in subtracting the tara its value to the nearest lb may be taken. On the same page he has a question in which there are two taras. The value is required of 27 sacks of cotton, which weigh in all 17061 lb, at the rate of 70 duc. 14 gr. 16 pic. the mearo, taking away for tara, on account of the sacks, 4 lb per sack, and from the remainder again taking away for tara 3 lb 6 oz per mearo (abbattendo di tarra, per conto di sacchi £4 per sacco, & del restante abbattendo anchora di tarra lire 3 oncie 6 per mearo), and 2 duc. 8 gr. per cent for messetaria. He begins the solution with the words "Per soluere questa, & altre simili, che hanno due tarre . . ."

This is the first example I have met with in an Italian book where the weight of the containing vessel is called tara, and almost the first in which it is mentioned in the question.*

Later on he gives two other questions in which the weight of the sacks is taken into account (nos. 8 and 9, pp. 152′, 153). In the first a mearo of rice is worth 6 duc. $3\frac{1}{2}$ gr., and the question is to find the value of 5326 lb (abbattendo di tarra per conto di sacchi, & di sporco £$5\frac{1}{6}$ per mearo), and $1\frac{1}{2}$ duc. per cento for messetaria. Here the weight of the sacks and the allowance for dirt or impurities are included as tara in a single percentage.

The next question is to find the value of 9 sacks of cotton, which weigh 5687 lb each, at the rate of 70 duc. $14\frac{1}{2}$ gr. the mearo (abbattendo prima di tarra per li sacchi £4 per sacco, & del restante anchora tarra £$3\frac{1}{2}$ per mearo, & di messettaria ducati 2 gr. 8 per cento). Here as in the question on p. 103 the tara for the sacks and the percentage tara are separated.

On pp. 153, 153′ Tartaglia gives two questions, relating to skins, in which an extra number of skins is given, but the word tara is not used. In the first 100 skins are worth $18\frac{1}{2}$ ducats, and 8 more are added as a gift (donandone sopra pelle 8. per cento), and in the second 5697 skins cost $34\frac{2}{3}$ ducats per mearo, and by custom 60 skins per mearo are given (per vsanza se ne dona sopra pelle 60 per mearo).

Remarks on the use of the word tara by Italian and German writers, §§ 124–125.

§ 124. The early Italian writers are consistent in their use of tara, which was a deduction from the gross weight agreed

* Sfortunati gave an example in which a deduction was made for sacks cordage, etc. (See § 118).

upon between buyer and seller on account of impurities, or for any other cause, and was expressed as a percentage on the gross weight. The word was not used for the deduction made for the weight of the sacks, casks, etc. Sfortunati makes a deduction for sacks, cordage, etc., but he does not call it tara (§ 118). Tartaglia, however, in his *Trattato* of 1556, has questions in which there are two taras, the first being for the weight of the sacks (§ 123). The general use of the word tara is as an equivalent to waste, and it is not paid for.

The word dono, or some word which expresses giving, is used when the abatement is made by adding an extra amount to the 100 lb of goods, the whole to be paid for as 100 lb, or adding an extra number to the 100 articles, to be paid for as 100*: so that if r is the addition the deduction is made in the proportion of $100 + r$ to 100. Tara is not used in this sense.

Fusti only occurs in connection with cloves. It was allowed for either by a simple rule (§§ 111, 114–116, 118, 122), or was wholly paid for at a lower rate (§ 116), or, after a certain small proportion had been allowed to be counted as pure cloves, was paid for at a lower rate (§ 122).

§ 125. Passing now to the German writers, we notice that Widman often makes a deduction for the wood of the containing vessel, which is not usual in the Italian books: that he uses the word fusti correctly (*i.e.* it is an inferior substance mixed with the cloves, and which has to be paid for): and that he gives a question relating to saffron in which there is absolute waste (unreyn) that has not to be paid for. Grammateus is the first to use the word tara, and he attributes to it its true meaning of waste: each pound 'holds' so much tara, and the tara is not paid for: it corresponds to Widman's 'unreyn' in the saffron question. In subsequent German books tara is the weight of the containing vessel or receptacle; but even when expressed as a percentage *auf* or *in* the centner it represents only this weight, and does not include a deduction for impurities or any other cause. The distinction between *auf* and *in* the centner was clearly convenient, and was (I think) an improvement on tara and dono, for dono was in principle also a tara, though differently expressed and slightly different in amount.

As a rule fusti received its true meaning of inferior goods mixed with 'pure' goods, and was generally used in connection with cloves; but at length its proper signification became confused with the deduction for the weight of the receptacle,

* The addition is sometimes made to 1000 lb or 1000 articles.

and it was used where we should have expected tara. Koebel seems to have regarded fusti as referring to the weight of the containing vessel as much as to impurities, and Spenlin almost restricted it to the weight of the containing vessel, though 'thara für ausswurff' occurs also under the heading of Fusti.

In the German Arithmetics the whole of the fusti in cloves was always paid for at a lower rate than the pure cloves, and I have not met with a single question in which the Italian practice of regarding a certain portion as waste and paying for the rest as pure cloves was followed.

The supposed mention of signs of addition and diminution by Peurbach, § 126.

§ 126. Reference has already been made* to signs of addition and subtraction which were mentioned by Peurbach. Drobisch considered that these signs were not + and −, but merely signs that were left to the reader: but Treutlein (partly influenced by a manuscript which was afterwards found to be of later date than he had supposed) took the contrary view, and considered that they referred to + and −. As Peurbach died in 1461, any reference by him to signs for addition and subtraction, whatever might be the signs he had in his mind, would be of great interest; and I have therefore examined with some care all the editions of Peurbach's Algorithmus to which I had access, in order to ascertain what evidence they afforded that he used, or recommended the use of, signs for addition or subtraction. I found the investigation less simple than I had expected, on account of the different titles under which the Algorithmus appeared, and some variations in the text. It spite of these differences there was a close agreement in the text of the editions up to 1536, all of which seem to have been derived from a manuscript first printed by Winterburg at Vienna about 1500. But in 1536 a much larger Arithmetic, having Peurbach's name as that of the author, was published at Wittenberg; this treatise contained not only the earlier work relating to integers, but also a long continuation relating to fractions, the rule of three, the rule of false, proportions, etc. It is in the explanation of the rule of false in this work that the 'signum additionis' and 'signum diminutionis' occur. It is of course possible that at that late date a neglected manuscript of Peurbach's on arithmetic came to light and

* In the last note on p. 9.

was published: but it may be that Peurbach was not the author, and that the work belongs to a later period.

I now proceed to describe the different editions of Peurbach's Arithmetic which I have seen, including that of 1536 (§§ 127—136). The discussion of the whole subject follows and occupies §§ 137—151.

The Vienna Algorismus (*c.* 1500), *denoted by* Al_1, § 127.

§ 127. The earliest version of Peurbach's Algorithmus that I have met with (and which is certainly a very early one) has the title "Opus Algorismi Iocūdissimū Mgr̄i Georgij peurbachij Wieunensis (p̄ceptoris singularis Mgr̄i Ioannis de monte regio) sacreque mathematice inquisitorte suptilissiō sūma cū vtilitate editū".* The book consists of six leaves, the first of which has merely the word 'Algorismus' on the recto: the title quoted above is at the top of the recto of the second leaf. The first paragraph or chapter has no special heading, but the words in the first line 'Numeri propositi rep̄ntatōnem' are printed in larger letters. The other chapters have special headings, 'De Additione', 'De Subtractione', 'De Mediatione', 'De Duplatione', 'De Multiplicatiōe', 'De Diuisione', 'De Progressione', following which, but without a special heading, is rather more than a page on the extraction of the square root, beginning with the words "Cuiuscunque nūeri quadrati vel maximi quadrati sub numero proposito cōtenti radicē quadratā extrahere", and ending with the words 'accipere potes'; followed by "¶Finis Algorismi Magistri Georgij de Peurbach". After this Finis there are several paragraphs, occupying about two pages, with separate headings, 'De regula aurea siue de tre', 'Secunda Regula que societatis vel mercatorum appellatur tali declaratur exemplo', 'Sequuntur nūc enigmata quorum primū est istud', 'Aliud enigma', 'Aliud enigma', 'Aliud', 'aliud'. The colophon is "Impressum Vienne per Ioannem Winterburg". The copy I have described is in the British Museum Library. The date is given in the catalogue as *c.* 1500, which is probably correct. This edition (by Winterburg of Vienna) will be referred to as Al_1.†

It will be noticed that there are two misprints in the title: Wieunensis for Wiennensis, and inquisitorte for inquisitore. Except in the first paragraph (where the initial letter is very

* In the titles and other quotations I expand those contractions which cannot be reproduced for want of the requisite type.

† This edition is *13600 in Hain. See § 148.

large) spaces have been left for the initial letters, which have not been inserted, and in two places (under 'De Mediatione' and 'De Duplatione') spaces have been left for the fraction $\frac{1}{2}$ which has not been inserted.

Other copies of this edition are mentioned in § 148.

Another Vienna edition, denoted by Al_2, § 128.

§ 128. In the Cambridge University Library there is another edition with almost the same title, and by the same printer, Winterburg. The title is "Opus Algorismi Iocūdissimū Mgr̄i Georgij peurbachij Wiennēsis (p̄ceptor singularis Mgr̄i Ioannis de monte regio) sacreque mathematice inquisitore sb̄tilissiō sūma cu vtilitate editū". This title differs from that of the book just described in the correction of the two misprints, and in having 'preceptor' instead of 'preceptoris'. The colophon is "Impressum Vienne per Ioannem Winterburg".

The whole type has been reset, but the pages are the same. The spaces for the initial letters are still left blank, but the fraction $\frac{1}{2}$ has been inserted where there were blank spaces in Al_1.

This is presumably a later edition than Al_1, but I cannot assign any probable date to it. It will be denoted by Al_2.

A comparison between Al_1 and Al_2 is given in § 137.

The Leipzig edition of 1503, § 129.

§ 129. In 1503 an edition of the Algorismus was published at Leipzig, which differs from the Vienna edition Al_1 in portions of the text, by the addition of an example to each rule, by the insertion of a chapter on cube root, and by the inclusion as part of the Algorismus of the paragraphs that follow 'Finis...' in Al_1. The title is almost the same as in Al_1, except that the final word 'editum' is replaced by "exemplis ac cubice radicis extractione alleuiatoque procedendi modo nuper digestum".* The colophon is "Et tantum de hoc opere Algoristico Anno Christi Hiesu Millesimoquingentesimotertio per Baccalariū Martinū Herbipolen̄ impresso".

In this edition the title occurs both on the recto of the first leaf and at the top of the recto of the second leaf, the heading

* The complete title is "Opus Algorithmi iucundissimū Magistri Georgij Peurbachij wiennensis (preceptoris singularis Magistri Ioannis de Monteregio) sacreque Mathematice inquisitoris subtilissimi summa cū vtilitate exemplis ac cubice radicis extractione alleuiatoque procedendi modo nuper digestum". This is the earliest dated edition I know of in which the change has been made from Algorismus to Algorithmus.

'Numeratio' is prefixed to the first paragraph; the other headings are 'Additio', 'Subtractio', etc., instead of 'De Additione', 'De Subtractione', etc.; and the heading 'Radicum extractio quadrata' is supplied to the long paragraph on square root, separating it from the heading 'Progressio'. This is followed by 'Radicum extractio cubica'. Then come 'Regula aurea sine detre', 'Regula Societatis', and three enigmata, all of which were placed after the end of the Algorismus in Al_1. A very important addition is that an example is given after each rule; thus as an example of addition 486 is added to 5975; as an example of division 59078 is divided by 74; under square root the root of 767376 is extracted, and similarly for the other rules.

As partly indicated in the title, this is the Algorismus Al_1 revised, enlarged, and provided with examples, which render it much more complete and useful. It is to be presumed that Al_1 represents the Algorismus as left by Peurbach; and that the editor of this edition has included what was not Peurbach's is shown by the occurrence in the text of the rules and enigmata which follow "Finis Algorismi Magistri Georgij de Peurbach" in Al_1. It may be noted that in this edition all the initial letters are missing, spaces being left for them. The two fractions $\frac{1}{2}$ for which blanks were left in the 1500 edition are inserted.

The Nuremberg edition of 1513, § 130.

§ 130. In 1513 an edition was published at Nuremberg which resembles Al_1. The title is "Georgij Peurbachij. Mathematici omniū acutissimi, institutiões in Arihtmeticam: cum alijs tum in primis adulescentibus necessarie". There is a preface headed 'Ioannes Marius Rhetus Studiosis', and then comes 'Sequitur Opusculum Peurbachij doctissimi'. The edition Al_1 is followed (there being no heading for numeration), and the treatise ends with 'accipere potes' (the portion relating to square root being left, as in Al_1, under the heading 'De Progressione'), after which is "Finis Algorithmi Magistri Georgij Peurbachij". Then come the 'De regula aurea...', the 'Secūda Regula...', and the five enigmata as in Al_1, after which other puzzle questions are added having the headings 'Sequitur ingeniosa ratio inuestigandi: calculo annulum: quem quis e cõuiuis...', 'Alius modus idem inueniēdi a Vadiano supostus', followed by 'Finis'. The colophon is "Nurnberge impressit Iohannes Weyssenburger Sacerdos. Anno. 15.13. Die vero xiiii mensis Aprilis".

Thus this edition follows Al_1 both in the portion attributed to Peurbach and in the rules and enigmata which follow it: and there are also two other puzzle questions relating to the detection of a concealed number. Where blanks were left in Al_1 for the insertion of $\frac{1}{2}$ the text runs right on in this edition, without either the space or the $\frac{1}{2}$, making the sentences unmeaning. It will be seen in § 148 that this edition is a reprint of a Vienna edition of 1511.

Tannstetter's edition (*Vienna*, 1515), § 131.

§ 131. In 1515 Tannstetter published at Vienna in one volume* five arithmetical works, one of which was Peurbach's Algorithmus. It appears under the title "Opusculū Magistri Georgij Peurbachij doctiss.", and is printed as in Al_1, ending with the words 'accipere potes', which conclude the account of square root. Then comes "Finis Algorithmi Magistri Georgij Peurbachij". The heading of the pages is "Algorithmus Peurbachij".

In the two passages where $\frac{1}{2}$ should have been inserted the text runs on without it as in the 1513 edition, and other small errors which were corrected in the Winterburg edition Al_2, and in the 1503 edition, are left unaltered (§ 138). It is surprising that Tannstetter should have reprinted the 1500 edition without any attempt at editing it, as is shown by the fact that not even the $\frac{1}{2}$'s are inserted.

The Wittenberg edition of 1534, §§ 132–133.

§ 132. In 1534 an edition was published at Wittenberg, having the title "Elementa arithmetices algorithmvs de numeris integris auctore Georgio Peurbachio. de nvmeris fractis, Regulis communibus, & Proporcionibus. Cum prefatione Philippi Melanchthonis. M.D. XXXIIII". The colophon, which occurs on E vii' is "Impressvm Vitebergæ per Iosephvm Clvg. Anno M.D. XXXIIII". At the head of the first page (after Melanchthon's preface) there is the title "Opvscvlvm magistri Georgii Pevrbachii doctissimi".

Except that the paragraph on square root has a heading, the algorithmus is given almost exactly as in Al_1 (the $\frac{1}{2}$'s being omitted and the text running on) up to 'accipere potes'.

* The title of the book is "Contenta in hoc libello. Arithmetica communis. Proportiones breves. De latitudinibus formarum. Algorithmus. M. Georgij Peurbachij in integris. Algorithmus Magistri Ioannis de Gmunden de minucijs phisicis". The dedication begins "Georgius Tannstetter Collimitius: artium et Medicine doctor: et Mathematice in studio Viennensi professor ordinarius . . . ." The colophon is "Impressum Vienne per Ioannem Singrenium Expensis vero Leonardi & Luce Alantse fratrum Anno domini. M.CCCCC.XV. Decimonono die Maij."

These words end the recto of B v (misprinted A v) and on the verso under the heading 'Lectori' there is a statement* that what precedes is by Peurbach, but that he either did not complete his compendium, or intended it only for boys, as all that was published in the Vienna edition has been faithfully transcribed. A sequel has therefore been added to complete the treatment of the subject. This consists of a 'Secunda pars' relating to fractions, a 'Tertia pars' containing the rule of three and the rule of false, with many examples, and a 'Quarta pars', entitled 'De proportionibvs ex elementis Evclidis, per Ioannem Vogelin.'

§ 133. Although this continuation had no connection with Peurbach, and there is no indication as to who was its author, it may be noted that in the rule of false the signs + and – are used, and placed outside the errors, which are written under the positions: thus (on E ii')

36 48

– 11 2 +

13

The signs are defined (on D iii') in the words "Si maior, signabis illum hac nota + Si minor illa –".

The Wittenberg edition of 1536 (*in which signum additionis and signum subtractionis occur*), §§ 134–136.

§ 134. Two years later, in 1536, another edition by the same printer appeared, in which a long continuation was appended to the original Algorithmus, the whole being described as the work of Peurbach. It is in this continuation (in the explanation of the rule of false) that the words 'signum additionis' and 'signum subtractionis' occur.

The title of the book is "Elementa arithmetices. algorithmvs de nvmeris integris, fractis, Regulis communibus, & de Proporcionibus. Autore Georgio Peurbachio. Omnia recens in lucem edita fide & diligentia singulari. An. M.D. XXXVI.

* This statement is: "Hactenus Peurbachius, qui aut non absoluit ceptum compendium, aut pueris tantum conscripsit quos in hac prima numerorum tractatione tantum exercere uoluit, bona enim fide quæ in exemplari Vienensi ædita sunt transcripsimus. Adiecimus autem id quod erat reliquum, prodest enim in scholis integram huius partis Methodum tradere, ut adolescentes in his numerorum principijs bene exercitati, ea facilius assequantur quæ & in altera Arithmetices parte et in reliquis omnib. Mathematicis disciplinis traduntur. Spero autem ex hac epitome summã huius partis facile cognosci atq; intelligi posse, si usus accesserit, qui omniũ Magistrorum præcepta longe superat".

Cum præfacione Philip. Melanth." The colophon (on G vii) is "Impressum Vitebergæ per Iosephum Klug. An. M.D. XXXVI."

The whole treatise runs on uninterruptedly, forming a complete work, and the continuation is quite distinct from that which was appended to the 1534 edition. After the words 'accipere potes', which end the original Algorismus Al_1, there is added another paragraph on square root in which a method of extraction or verification is explained, followed by an example (the first example which occurs in the work). Then comes (on B vi) a chapter on cube root, of which no example is given. On B viii′ there is a principal heading, 'Algorithmvs de minvciis', which seems to apply to the rest of the book, although the rules for the treatment of fractions end on D i. Then under the heading "Qvasdam regvlas ad inveniendum numerum ignotum per notos sibi proporcionatos subiungere" there are two pages on proportions, giving the rules by which, when two out of three numbers in continued proportion are known, the third can be found, and by which, when three numbers out of four in simple proportion are known, the fourth can be found. Then on D ii, under the heading 'De regvlis', the author begins with the paragraph "In Elementis primis Arithmeticæ practicæ stetimus hucusque nunc ad res scitu digniores animum applicabimus, quas nisi nacti fuerimus inaniter diem transiuit omnis opera. Quid enim literas uidisse proderit, si dictiones contexere nequeas? Quanti facies instrumentū, si non eius calleas usum? Adesto igitur æquo animo & regulas accipe, ut pernoscas quid spei sit reliquum post hac quod ex Enigmatibus afferam". This introduction is followed by two paragraphs, which are almost a repetition of the rules just given for finding a third or fourth proportional,* and the next paragraph explains that if a number is the sum of several others, then any other number can be expressed as a similar sum by means of a simple proportion.†

§ 135. He then passes to the rule of false, of which the heading (on D iii′) is "Nvnc ad regvlam positionis falsæ, quam

* On D i and D i′ 'proportionalitas' is 'continua' or 'discontinua', but in the repetition on D ii and D ii′ numbers are proportional 'continue' or 'incontinue'. This seems to suggest that a new departure is made with 'De regvlis' on D ii. The use of a letter to denote an unknown on D ii′ is noticeable: as an example of finding a third number in continued proportion the author writes "in exemplo tres sunt numeri 4. 6. et A. ignotus quem tibi notum dari uelles, multiplica terminum cōmunem scz 6. in se fiunt 36. quæ diuide per 4. exeunt 9. numerus scilicet A prius ignotus". The rule is then given that if the extremes are known the mean is found as the square root of their product.

† This is considered a separate rule, being introduced by the words "Aliam Regulam quæ prioris filia ccusetur tibi dabo".

Arabes Stahaim appellant, attentus sis uolo". In describing the rule, in reference to the first position he writes "Quod si non dabit ueram solutionem, dabit tum aliquem numerum uel minorem uera [solu]cione uel maiorem ea. Si minorem, subtrahe eum à uera solutione, & residuum uocabitur error diminutus. Si autem maiorem, ab eo subtrahe ueram solutionem, & residuum appellabis errorem additum. Huiusmodi ergo errorem primæ positioni subscribe cum signo denotante ipsum fuisse additum uel diminutum". A second position is then to be taken, and he proceeds "Si autem non occurret uera solutio, nota errorem ut prius, hunc errorem pone sub sua positione cum signo additionis uel diminutionis". He then gives the rule for deducing the true result from the positions and errors; the errors are referred to as additi or diminuti, but the signum is not mentioned.

Only one example is given. Two companions wish to buy a horse, of which the price is 10 florins. The first will have enough to buy it if the second gives him half of what he possesses: the second will have money enough if the first gives him $\frac{1}{3}$ of what he possesses. Suppose the first has 3 fl, then the second must have 14 fl, and therefore with $\frac{1}{3}$ of what the first has he has 15 fl with which to buy the horse, and "erit error additus 5. quem scribe sub tribus cum signo additionis." He then supposes that the first has 4 florins, and gives the direction "operare sicut prius, tandem ueniet error additus 3. & $\frac{1}{3}$ hunc subscribe suæ positioni cum additionis signo."

After this he gives various problems under the heading (on D v) "Nvnc ad enigmata varia descendamus". The last of these (on E i'), which is similar to the example under the rule of false which has just been described, is: three companions wish to buy a horse, the price of which is 100 fl. The first will have enough to buy it if the second gives him $\frac{1}{2}$ of what he has: the second will have enough if the third gives him $\frac{1}{3}$ of what he has: and the third will have enough if the first gives him $\frac{1}{4}$ of what he has. The problem is solved by the rule of false exactly in the same way as the previous question relating to the purchase of a horse. The first position is 70 fl (for the money of the first) which gives $137\frac{1}{2}$ fl for what the third will have to buy the horse with: "errauit ergo positio prima in $37\frac{1}{2}$ additis, scribe ergo $37\frac{1}{2}$ sub 70. scilicet positiōe cōmuni prima cum signo additionis." For the second position he takes 80 fl which gives 200 fl for what the third will have for the purchase of the horse. "Vnde constat positionem secundam errasse in 100. additis, scribe ergo 100. sub 80. positione scilicet secunda cum signo additionis."

The book ends on G ii′ with some examples of the different kinds of proportions: then follows (G iii to G vii) the 'De proportionibvs ex elementis Evclidis, per Ioannem Vogelin', which had already appeared in the 1534 edition, the colophon being on G vii.

§ 136. If this continuation was really written by Peurbach it would show that in the rule of false he used signs to indicate whether the error was additus or diminutus, and that these signs were called the signum additionis and the signum diminutionis: but it seems uncertain whether this continuation is correctly ascribed to Peurbach. The natural inference from the two editions of 1534 and 1536 would be that in the former year Klug printed Peurbach's original Algorithmus, supposing that this was all he had written (in his own words, transcribing the Vienna edition faithfully) and appended to it a continuation by a contemporary writer so that the whole might form an adequate introduction to arithmetic; but that subsequently a manuscript of a more complete arithmetic wholly due to Peurbach himself came to light and that this manuscript 'edita fide & diligentia singulari' formed the *Elementa Arithmetices* of 1536.

A circumstance in favour of its genuineness is that it was published in conjunction with Voegelin's Geometry*, and that Voegelin was a distinguished professor at the University of Vienna, where Peurbach had lectured: and it does not seem likely that he would have allowed an arithmetic having Peurbach's name to be attached to a work of his own unless he believed it to be authentic.†

Comparison of the two Winterburg editions Al_1 *and* Al_2, § 137.

§ 137. Before considering further the authenticity of the portion of the arithmetic which was first published in 1536, it is convenient to compare the different editions of the portion

* The two books were issued together, the joint title being "Elementa geometriae ex Evclide singulari prudentia collecta à Ioanne Vogelin professore Mathematico in schola Viennensi. arithmeticae practicæ per Georgium Peurbachium Mathematicum. Cum præfacione Philippi Melanthonis". Melanchthon's preface and a page of verses occupy seven leaves, then the Geometry begins on B i, without a separate title-page, and ends on F vii with the colophon "Impressum Vitebergæ per Iosephum Klug. M.D. XXXVI." Then comes the title-page of Peurbach's *Elementa Arithmetices* on A i. The Arithmetic ends on G ii′, and is followed by Voegelin's proportions from Euclid, the colophon being on G vii. The account of the Venetian reprint (§ 150) given in *Rara Arithmetica* (p. 53) shows that in that edition both works were issued together in the same manner.

† A volume of manuscripts in the Vienna library, Codex Vindob. 5277, belonged to Voegelin, most of the manuscripts and the index being in his handwriting (Wappler, note to p. 3 of his *Programm* referred to in § 152).

ending with 'accipere potes' (in the chapter on square root), the genuineness of which may be assumed.

In the first place it is worth while to notice some differences between the two undated editions printed at Vienna by Winterburg, which have been denoted by Al_1 and Al_2 (§§ 127, 128). To the former the date 1500 has been assigned, and it may be even earlier.

Besides the misprints in the former, referred to in § 127, and some slight variations, the following differences occur: In l. 18 of the chapter on addition, an 'in' is inserted in Al_2 which does not appear in Al_1, the words in Al_1 being 'aūt isto opere', and in Al_2 'aūt ī isto opere'. In l. 7 of the chapter on subtraction, Al_1 has 'scribe tibi' and Al_2 'scribe ibi': and in l. 8 Al_1 has 'qm̄ min⁹ a mīori', and Al_2 'qm̄ mai⁹ a mīori'. In l. 16 of the chapter on mediation and l. 4 of that on duplation a space has been left for the fraction $\frac{1}{2}$ in Al_1, but the $\frac{1}{2}$ has been inserted in Al_2. Of these differences the 'in' is trifling, but 'minus a minori' is clearly a slip which needs correction, and of course the fraction $\frac{1}{2}$ (left out in Al_1 presumably for want of type) should be inserted.

Variations in the different editions, §§ 138–140.

§ 138. Passing now to all the editions which I have been able to examine, viz. Al_1 (1500?), Al_2 (uncertain date), 1503 (Leipzig), 1513 (Nuremberg), 1515 (Vienna, Tannstetter's edition), 1534 (Wittenberg), 1536 (Wittenberg), it will be seen that they fall into groups in regard to certain verbal variations which occur.

These variations are as follows; the editions printed at Vienna by Winterburg being denoted by Al_1 and Al_2 as before, and the other editions by their dates.

In 'De additione', 'autem isto opere' occurs in Al_1, 1513, 1515, 1534, and 'autem in isto opere' in Al_2, 1503, 1536.

In 'De subtractione', 'scribe tibi' occurs in Al_1, 1513, 1515, 1534, 'scribe ibi' in Al_2, 1503, and 'ibi scribe' in 1536: 'quoniam minus a minori subtrahi non consueuit' occurs in Al_1, 1513, 1515, 1534, while minus is replaced by maius in Al_2, 1503, 1536: also 'obseruare fiat oblatum ciffre' occurs in Al_1, Al_2, 1513, 1515, but it is 'obseruare fiat ablatio' in 1534, and 'considerare fiat ablatio' in 1536.

In 'De Mediatione' the sentence 'Et medietatem sub tali digito aut cifra inferius scripto' occurs in Al_1, Al_2, 1513, 1515, 1534, but 'scripto' is replaced by 'scribes' in 1503, 1536.

In 'De Mediatione' and in 'De Duplatione' the $\frac{1}{2}$ is omitted in Al_1, a space, however, being left for it; and it is omitted without any space being left in 1513, 1515, 1534. The $\frac{1}{2}$ is inserted in Al_2, 1503, 1536.

The omission of $\frac{1}{2}$ in mediation and duplation is serious, as without it the sentences are unintelligible. In the former the learner is directed, if the number to be halved ends with an uneven number, 'post finem numeri aliquo spacio interiecto scribere; ut sic $\frac{1}{2}$ talis vnitatis medietatem', and in the latter the rule is 'Si tamen haberes extra ordinem $\frac{1}{2}$ pro tali duplando adderes primis prius in dextera parte vnitatem'.

§ 139. The preceding comparison shows that Al_1, Al_2, 1513, 1515, 1534 are substantially the same and that they have presumably been all derived from Al_1 or some equivalent text.

The principal differences are that $\frac{1}{2}$ is inserted in Al_2, and that 'oblatum ciffre' becomes 'ablatio' in 1534; but obvious emendations, such as maius for minus or scribes for scripto, have not been made.

Among the editions in which there has been no serious change Al_2 is the best. The absence of '$\frac{1}{2}$' in mediation and duplation in so many editions, including Tannstetter's, shows how little care was taken in the re-issues.

The Leipzig edition of 1503 seems to have been derived from Al_1 (*i.e.* not from an independent source), but to have been enlarged and improved by a competent editor: so that it is not wholly due to Peurbach. It is noticeable that some of the changes made in this edition occur also in that of 1536.

It was mentioned in § 76 that Lacher's Algorithmus Mercatorum up to the end of 'De Divisione' was a reprint of Peurbach's Algorithmus, and, as one would expect, it was the Leipzig edition of 1503 which he copied. All the divergencies from Al_1 which occur in the latter are reproduced in Lacher's Algorithmus.

§ 140. Coming now to the edition of 1536 there are numerous differences of wording from the Vienna editions, many of them of slight importance, but indicating either that this earlier portion has been revised or that the text has been derived from another source. Thus in the first chapter (on numeration) 'significat figuram primariam ipsius impositionis. In secundo vocatur decies tantum' becomes, in the 1536 edition, 'significat secundum primariam ipsius impositionem. In secundo uero decies tantum'. In the first sentence of

'De Mediatione' viz. 'Numerum quencunque mediare', the last word is replaced by 'dimidiare'. The first sentence of 'De Multiplicatione', viz. 'Numerum quencunque multiplicare', is amplified by the insertion of 'per quencunque' before multiplicare, and this is followed by a new sentence 'Multiplicare non est aliud', etc. In 'De Divisione' the sentence 'In his autem omnibus speciebus . . .' becomes 'Præterea in his omnibus speciebus . . .' In 'De progressione' in 'iunge etiam primum ultimo' the word 'locum' is introduced after 'primum'; and three lines before 'accipere potes', which ends the Algorismus in the earlier editions, in 'si tale superfluerit dempseris', the word superfluerit is replaced by superfluum. There are also other such changes.

References to Peurbach's Algorithmus by Drobisch, Gerhardt, Treutlein, Unger, Tropfke, and Cantor, §§ 141–146.

§ 141. Drobisch* seems to have been the first to direct attention to Peurbach's mention of the signs of addition and subtraction. After pointing out that + and − were first used by Widman, but in such a way as to suggest that they were already known in Germany, he states that he has failed to find them in Peurbach or Regiomontanus: but in connection with the former he adds the note "In Peurbachii algorismo, ubi regula falsi exponitur, de signis additionis et diminutionis sermo quidem est, sed lectoris libero arbitrio relictum videtur, commoda signa sibi eligere."

It is evident therefore that Drobisch must be referring to the 1536 edition or a reprint, or, at all events, an edition having the same text.

§ 142. Gerhardt, on pp. 9–11 of his *Geschichte der Mathematik in Deutschland* (1877), gives an account of Peurbach's Algorismus (ending with square root), but he makes no reference to the continuation, or to Peurbach's allusion to the signs†.

He quotes Grammateus's statement that Peurbach's *Algo-*

* *De . . . Widmanni . . . compendio* (1840), p. 20.

† He describes the Algorismus in its original form, but states that later it was often enlarged by examples and additional notes. The example by which he illustrates Peurbach's method of division (viz. the division of 59078 by 798) is the one given in the Leipzig edition of 1503. In a note he gives the three titles 'Introductorium in Arithmeticam', 'Algorithmus de integris', 'Opusculum Magistri Georgii Peurbachii', referring to Aschbach (see § 149), and he also mentions Tannstetter's edition of 1515.

rismus was written for the young students of the high school at Vienna*.

§ 143. Treutlein, writing in 1879†, adopts Drobisch's opinion that + and − were already in use in Germany by Widman's time, but, differing from him, considers that they were the signs referred to by Peurbach. His words are "Ich meinerseits finde eine Bestätigung hiefür, Drobisch's Ansicht entgegen, auch bei Peurbach (†1461): wo dieser die Regula Falsi erklärt, verlangt er, dass man eine gewisse Zahl '*cum signo denotante ipsum* (*numerum*) *fuisse additum uel diminutum*', oder an einer andern Stelle, dass man sie anschreiben solle '*cum signo additionis uel diminutionis*', wobei freilich Peurbach selbst die Zeichen nicht gebraucht; mir scheint es einem Zwange gleichzukommen, wenn man hierin den Gedanken an den Gebrauch der Zeichen + und − nicht annehmen wollte".‡ It would seem that Treutlein had not himself seen Peurbach's Algorismus.§

§ 144. Unger in his *Die Methodik*‖ (1888) makes no reference to the mention of the signs by Peurbach. On p. 35 he gives the title of the edition of 1536 and describes its contents. At the end he refers to an earlier edition in the words "Wildermuth nennt einen aus sieben Quartblättern bestehenden Algorithmus Peurbachs, welcher 1505 gedruckt ist". On p. 25 he had said that from Grammateus we learn that the Algorithmus of Peurbach "gemacht sei für die Studenten der hohen schul zu Wien", and he remarks that it only contains the amount of knowledge that children of ten

* This sentence, which has already been quoted in § 46 (p. 35), is "Vnd diese regel beschreybt vns Maister Georgius von burbach jn dem lateinischen algorithmo, gemacht fur die jungen studenten der hoen schuel zu Wien". Grammateus's own book (no. 5° of § 42, p. 30) was "gemacht auff der lobliche̅ hohe̅ schul zu wienn".

† *Zeitschr. für Math. u. Phys.*, vol. xxiv., supp., p. 29.

‡ He was confirmed in his opinion by a manuscript printed by Gerhardt, in which + and − occur. (See the last note to § 13, p. 9). This manuscript has the heading *Regule Cose vel Algobre*, and is the first manuscript in the Vienna Codex 5277. An extract from it was published in the *Monatsberichte* of the Berlin Academy for 1870 (pp. 143–147) by Gerhardt, who assigned it to the middle of the 15th century. It has since been shown that the whole Codex belongs to the 16th century (Wappler, p. 3, note, of his *Programm*, referred to in § 152; Cantor, *Vorlesungen*, vol. ii., p. 240; Curtze, *Centralblatt für Bibliothekswesen*, Jahrgang 16, p. 290). Curtze (*l.c.*) found in the Munich Codex 19691 another manuscript of the *Regule Cose vel Algobre* bearing the date 1510, so that this treatise certainly is not later than the beginning of the 16th century.

The Vienna Codex 5277 has been already mentioned in the second note to § 136 (p. 98) as having belonged to Voegelin.

§ He refers to the Algorismus in vol. xxii. of the *Zeitschrift* (1877), supp., p. 11, but quotes Wildermuth with respect to its contents.

‖ "Die methodik der praktischen arithmetik in historischer entwickelung . . ." (Leipzig, 1888).

years of age now possess.* This is true of the Vienna and other early editions, but not of that of 1536, which is the only one described by Unger.

§ 145. Tropfke, on p. 132 of vol. i. of his *Geschichte der elementar-mathematik*, refers to Treutlein's account of the use of the words signum additionis, etc., by Peurbach, but expresses his doubts as to whether the words quoted by Treutlein were originally written by Peurbach, as they may have been the additions of a later editor. He mentions that they are wanting in the various editions which he has seen, and that he has never met with them.† His words are "Aber es ist ausserordentlich zweifelhaft, ob das Originalbemerkungen Peurbach's sind. Man kann annehmen, dass hier spätere Zusätze vorliegen, die ein Herausgeber des Peurbach'schen Buches sich erlaubt hat; einmal fehlen nämlich in verschiedenen Auflagen diese Bemerkungen gänzlich, dann ist die gebrauchte Redewendung die beliebte Ausdrucksweise in verschiedenen Rechenbüchern des beginnenden sechzehnten Jahrhunderts, als die Zeichen + and − längst benutzt wurden".

With reference to the last sentence, it may be remarked that we should certainly not expect the words signum additionis or signum diminutionis to be used for + and − in the rule of false unless the signs were well known (for the idea to be conveyed is that of more or less, and not addition or diminution). But their use in the 1536 edition of Peurbach is justified by the preliminary steps: first we have an 'error additus' or 'error diminutus' according as the result exceeds or falls short of the true value. Then comes a 'signum' to denote whether the error was additus or diminutus, and then a signum additionis and signum diminutionis.

§ 146. Cantor's account of Peurbach's Algorismus is given on pp. 180, 181 of vol. ii. of his *Vorlesungen*. He says it was

* "Von Grammateus erfahren wir, dass der Algorithmus M. Georgii Peurbachii, der etwa dasjenige arithmetische Mass von Wissen enthält, welches gegenwärtig zehnjährige Kinder besitzen, 'gemacht sei für die Studenten der hohen schul zu Wien'". Here, and also on p. 35, Unger leaves out the word 'jungen', which should be inserted before Studenten. Grammateus's statement could only refer to the Algorismus in its early form, as the book in which it occurs was published in 1518.

† "Verfasser hat verschiedene Ausgaben des Peurbach'schen Rechenbuches eingesehen, ohne die von Treutlein angeführten Bemerkungen zu finden." Thus Tropfke had not seen the edition of 1536, and therefore did not know the extent of the additional matter in this edition, viz. 74½ pp. out of 95. On p. 28 he quotes Unger's statement that Peurbach's Algorismus contains only what children of ten now know, which was true of the editions that Tropfke had seen.

first published at the end of the fifteenth century, perhaps in 1492, under the title "Opus algorismi jocundissimum" and often reprinted with this title, or "Institutiones in arithmeticam", or merely as "Opusculum Magistri Georgii Peurbachii". He states that like the Algorismus of Sacrobosco it relates only to integers, but that an edition superintended by Melanchthon and Voegelin, and printed at Wittenberg in 1536, contains an algorithmus de minuciis and an algorithmus de proportionibus ascribed to Peurbach, of which the first seems to be genuine, as it occurs in a Munich manuscript of the fifteenth century.* He then mentions that Peurbach seems to have fallen short of his predecessor Sacrobosco, in omitting cube root, although from the different editions which were all published at least 30 years after Peurbach's death a positive conclusion cannot be reached (wiewohl aus den unter einander verschiedenen Drucken, die ja alle mindestens 30 Jahre nach Peurbach's Tode erfolgten, ein sicherer Schluss nicht gezogen werden kann), and he refers in a note to Gerhardt's description of the Algorismus and Günther's description of an edition of 1503, in which cube root, the rule of three, etc. occur. The latter is presumably the Leipzig edition described in § 129.

Cantor's statement that the Algorithmus de minuciis in the 1536 edition occurs also in a Munich manuscript of the fifteenth century is important.

Editions of Peurbach's Algorithmus mentioned by bibliographers and others, §§ 147–150.

§ 147. I now mention editions of the Algorithmus which have been recorded or described by bibliographers and others.

In his *Versuch*† Khautz devotes most of chapter ii. (pp. 33–57) to Peurbach, giving a description of his works on pp. 45–57. The paragraph describing his writings on arithmetic begins "VI. Introductorium in Arithmeticam. Apfaltrer‡ schreibt: es sey hier im Jahre 1511, durch Hieronymus Binder, oder, wie sich dieser . . . mochte genennet haben, Victor,

* "allein eine durch Melanchthon und Voegelin besorgte, in Wittenberg gedruckte Ausgabe von 1536 enthält einen dem Peurbach zugeschriebenen algorithmus de minuciis und einen algorithmus de proportionibus, von welchen der erstere echt zu sein scheint, da er auch in einer Münchener Handschrift des XV. Jahrhunderts vorkommt."

† "Versuch einer Geschichte der Oesterreichischen Gelehrten . . . von Franz Constantin Florian von Khautz" (Frankfort and Leipzig, 1755).

‡ I have not been able to see Apfaltrer's work In Adelung's "Fortsetzung . . . zu . . . Jöcher's Allgemeinem Gelehrten-Lexico" (vol. i., Leipzig, 1784) the title is given as "Scriptorum antiquiss. ac celeberr. Vniversit. Viennensis Pars I. Wien, 1740".

herausgegangen.* Nachgehends ist diess Rechenbüchlein in gothischem Drucke zu Nürnberg in 4 ans Taglicht getreten:". The title and the colophon of the work are then given, and they correspond to the Nuremberg edition of the same date described in § 130. Finally the full title of the Wittenberg edition of 1536 is given. Mention is also made of the preface by Johann Marius Rhetus 'an die Kunstbeflissenen' in the former, and of the preface of Melanchthon and dedication to Justus Jonas in the latter.

It might be inferred from the first sentence that "Introductorium in Arithmeticam" was the title of the book printed at Vienna in 1511 by Vietor; but this is not so, for the title of the work, which is given by Denis (§ 148), is the same as that of the Nuremberg edition of 1513.

On p. 56 Khautz says that besides the books which he has mentioned under 'VI.' Apfaltrer attributed to Peurbach still another Arithmetic, viz. "Algorismus. Editus Viennae Austr. typis *Io. Winterburgii*, sine anno. habetur in Biblioth. Academica.", but he thinks that Apfaltrer is in error, and that the book is not Peurbach's. He proceeds "Denn, was war es nöthig, dass sich Peurbach im Abc der Mathematik verweilen, und zwo Anleitungen dazu verfertigen sollte?" He thinks it likely that the book was written by an unknown author, and gives an example of a wrong attribution in the case of another book. Apfaltrer, however, was correct, and the book referred to is one of the Winterburg editions.

Thus Khautz mentions four editions: a Winterburg Algorithmus, and those of 1511 (Vienna), 1513 (Nuremberg), 1536 (Wittenberg).

§ 148. Denis, in his *Wiens Buchdruckergeschicht*,† gives the titles of four editions of Peurbach's Algorithmus printed at Vienna, two by Vietor (1511, 1515), one by Singrenier (1520), and one by Winterburg (without date).

On p. 59 he describes the book of 1511 referred to by Apfaltrer and Khautz. The title is the same as that of the Nuremberg edition of 1513 (except for the spelling of Arith-

* The twenty headings under which Khautz describes Peurbach's writings are taken from a list in "Tabulæ Eclypsiū Magistri Georgii Peurbachij", which was published by Tannstetter at Vienna in 1514. In the introductory matter (on aa 3′) there is a brief life of Peurbach, followed by a list of his writings, which, it is stated, was made by Stiborius (the instructor of Tannstetter). The sixth title in this list is "Introductorium in Arithmeticam", which was adopted as a heading, as also were the other titles, by Khautz. But I do not think that Stiborius meant it for the title of a book, or that it was so understood by Khautz.

† "Wiens Buchdruckergeschicht bis M.D. LX. Von Michael Denis" (Vienna, 1782).

meticam), and the colophon is "Impressum Vienne Pannoniæ ab Hieronymo Philoualle.* Anno. M.D.XI. decima die Martii." He refers to the preface of Joh. Marius Rhetus, quoting the concluding words, and, after describing the Algorithmus, proceeds "Nicht vom Peuerbach sind die angehängte goldene, die Gesellschaftregel und einige räthselhaften Aufgaben, davon die letzte dem Vadian zugeschrieben ist".

It would therefore seem that the Nuremberg edition of 1513 was an exact reprint of the Vienna edition of 1511.

On p. 128 he describes another edition of the Algorithmus printed by Vietor in 1515. The title is "Algorithmus Georgii Peurbachii Mathematici omniū acutissimi nō tā utilis quam necessarius.": this edition also contains the same additions (rule of three, etc.). The colophon is "Per Hieronymū Vietorē Viennæ Austriæ 19 die Octobris. Anno 1515."†

He also (p. 215) describes another edition, with the same title, printed by Singrenier in 1520, and having the colophon "Per Joannem Singrenium, Viennæ Austriæ. Mense Feb. 1520."

Among the undated books (on p. 313) is the Algorismus, described in §127 and denoted by Al_1. It is identifiable by the misprints 'Wieunensis' and 'inquisitorte'. Denis's comment is "Vielleicht die ungeschickteste Aufschrift unter allen unsern Producten". He finds it difficult to assign a date to it, but adds "doch wollte ich sie lieber in Winterburgers frühere Zeit setzen.‡

§ 149. Aschbach in his *Geschichte der Wiener Universität*§ (pp. 486 and 487) mentions various editions of Peurbach's Arithmetic, but all his information seems to have been derived from Apfaltrer, Khautz, Denis, and Hain, and not from an inspection of the works themselves. Thus he says that Peurbach's mathe-

* On p. vii Denis gives an account of Hieronymus Vietor, the printer. He used the names Philovallis, Doliarius, Vietoris, and also simply Hieronymus. Probably his German name was Binder. He first printed on his own account at Vienna in 1510.

† Denis notes that this book shows that Khautz was wrong in doubting whether Peurbach ever published an Algorismus, and that further confirmation of the publication is afforded by the Wittenberg *Elementa Arithmetices* of 1536 (in which 'algorithmus de numeris integris' is part of the title), and by the editions of 1520 and the undated edition by Winterburg, both of which he describes. He adds that the Algorithmus was omitted by Tannstetter in his list in the 'Tabulæ Eclypsium' because he regarded it as included under 'Introductorium in Arithmeticam'.

‡ This title is *13600 in Hain (*Rep. Bib.*, vol. ii., pt. ii.). Hain also gives (*13598 and *15999) the titles of two other editions without place, or date, or name of printer. The first of these contains three misprints, 'wieunensis', 'singulaıis', inquisitoris' (u for n in each case), but the second title is correct. They have Algorismus' alone on the first leaf, as in Al_1.

§ "Geschichte der Wiener Universität im ersten jahrhunderte ihres bestehens . . . von Joseph Aschbach" (Vienna, 1865).

matical works are his 'Algorismus oder Arithmetik' and his 'Einleitung in die Arithmetik'.

Of the Algorithmus he mentions the Vienna editions of 1515, 1520, and the undated editions: and of the 'Introductorium oder Institutiones in Arithmeticam' he mentions the editions of Vienna, 1511, Nuremberg, 1513, and Wittenberg, 1538, "durch Phil. Melanchthon herausgegeben, der das Buch unrichtiger Weise dem Justus Jonas zuschreibt."*

As we have seen, the Algorithmus and the Institutiones are the same work, and probably there was no edition with the word Introductorium in the title: but the edition of 1536, though including the Algorithmus, is a very much larger work.

§ 150. In *Rara Arithmetica*, Eugene Smith (p. 53) describes the Wittenberg edition of 1534, and also an edition published at Venice in 1539 which must be a reprint of the Wittenberg edition of 1536. The title is the same as that of the 1536 edition, but the colophon is "Venetijs Ioan. Anto. de Nicolinis de Sabio. Sumptu uero D. Melchioris Seffæ. Anno Domini M D XXXVIIII. Mense Iannario."

As in the case of the 1536 edition, the Arithmetic was issued with Voegelin's Geometry, the joint title-page given in *Rara Arithmetica* being the same (except for changes in a few letters) as in the 1536 edition. No reference is made to the difference between the Arithmetics of 1534 and 1539, which are treated as different editions of the same work.

Concluding remarks, § 151.

§ 151. Until further information is forthcoming with respect to the origin of the edition of 1536, I do not think that any certain conclusion can be reached. It might be supposed that Tannstetter would have published in his collection of works in 1515 the more complete Arithmetic of Peurbach had he known of its existence: on the other hand, Voegelin seems to have been concerned in the actual publication. Without further knowledge of the source from which the book of 1536 was derived, it seems best to regard it as not affording any trustworthy evidence of the use of signs of addition and subtraction in Peurbach's time.

* Aschbach is merely following Apfaltrer and Khautz, and so I think that 1538 is a misprint for 1536. I do not understand 'unrichtiger Weise', unless Aschbach understood Khautz to say that Melanchthon attributed the book to Justus Jonas: but zuschreibt here means dedicates.

Part III.

The German and Latin Algebras in Codex C 80 *of the Dresden Library,* § 152.

§ 152. I pass now to the consideration of the manuscript sources of information which Widman had at his command when writing his *Rechenung* (1489), and from which he probably derived his use of the signs + and −.

In 1887 Wappler printed, as an 'Abhandlung zu dem Programm des Gymnasiums zu Zwickau,'* a portion of a German Algebra and the whole of a Latin Algebra which are contained in the volume Codex C 80 in the Royal Library at Dresden. This volume, Wappler states, was originally in Widman's possession.†

The German Algebra in C 80, §§ 153–154.

§ 153. In the German Algebra,‡ the date of which is 1481, the cossic quantities are zall, dingk,§ zensi, chubi, wurzell von der wurzell, and there are signs for the first three, the fourth being denoted by chu, and the fifth by r|| von r. The words und and minner are used to connect quantities by addition and subtraction. Thus to multiply 4θ minner 5ϑ by 2θ minner 3ϑ, he says that 4θ by 2θ makes 8ʒ, that 3ϑ by 4θ makes 12θ minner, and that 5ϑ by 2θ makes 10θ minner, so that altogether it makes 8ʒ and 15ϑ minner 22θ.¶

§ 154. This example of multiplication is immediately followed by another, in which − is used in place of minner. "Aber 3ϑ − 2θ stund 6θ vn̄d 5ϑ so sprich 3ϑ stund 6θ macht 18θ Nu sprich 3ϑ stund 5ϑ macht 15ϑ Darnach mache 2θ stund 6θ macht 12ʒ − vn̄d mach 2θ stund 5ϑ 10θ − als 18θ

* "Zur Geschichte der deutschen Algebra im 15. Jahrhundert. Abhandlung zu dem Programm des Gymnasiums zu Zwickau von Oberlehrer Dr. Wappler" (Zwickau, 1887).

† *Programm*, p. 9. "Als den frühesten Besitzer des Dresdner Codex C 80 habe ich Johann Widman von Eger ermittelt."

‡ *Programm*, pp. 3–5. When Wappler published his *Programm* in 1887 he had not deciphered the inscription containing the date. This he gave subsequently in the *Zeitschrift für Math. u. Phys.*, vol. xliv., supp. vol., p. 539 (1899). As this periodical will be frequently referred to in this Part it will be quoted simply as *Zeitschrift*.

§ The signs for zall and dingk (which seem to be variations of the letter *d*) I replace by ϑ and θ, to which they have some resemblance. The sign for zensi is ʒ.

|| This sign, which I replace by a simple r, resembles r with a loop or flourish attached.

¶ The multiplication of 5ϑ by 3ϑ, both of which follow 'minner', is not explicitly given, but 15ϑ appears as their product in the final result.

vnd 15.9 minner 12ʒ vn̄d min̄er 10θ". Then, taking 10θ from 18θ, there remains '8θ vn̄d 15.9 min̄er 12ʒ'.

In these examples minner and – are placed after the quantity to which they refer when this quantity stands alone (*i.e.* so that there is no other quantity from which it can be subtracted). In the second example the sign – is used in the statement of the question, and in the working, but not in the final result.

Although Wappler gives only a portion of this German Algebra, I think we may infer that this is the only place where the sign – occurs.

The Latin Algebra in C 80, §§ 155–156.

§ 155. On pp. 11–30 of his *Programm*, Wappler prints the whole of the Latin Algebra. There is nothing in the manuscript to show its date, but evidence will be given (§ 158) which proves that it cannot be later than 1486. It consists of the 24 rules of algebra and of a great number of examples and problems illustrating their application. The names are numerus, cossa and radix, census, cubus; but the unknown is usually referred to as res. The cossic symbols resemble those that afterwards became general.*

The signs + and – are freely used throughout. The sign + first occurs on p. 13 in a list of 19 rules, the eighth of which is "1œ + 2ʒ equantur 15ψ. ψ est 3". It occurs also in nine more rules in this list, and in three places in the text on the same page. The sign – also occurs for the first time on this page in the expression "radix aggregati – medietate ψ est valor rei". In the following seventeen pages the signs + and – are continually used just as at present. They occur not only in simple expressions such as $\frac{1}{2}\psi + 2\frac{1}{2}\phi$ (p. 18), $63\psi + 1$ʒ (p. 22), 6ʒ + $6\phi - 12\psi$ (p. 26), but in fractions such as

$$\frac{10\psi - 1\mathfrak{z}}{120 - 24\psi} \text{ (p. 25)}, \quad \frac{480\psi + 480\phi}{128\mathfrak{z} + 256\psi} \text{ (p. 27)}.$$

In one question (p. 22) the sign – is used where we should have expected the word rather than the sign. In this question a person buys a certain number of ells of cloth for 60 florins, and if there had been three more for the 60 florins, an ell would have cost 1 florin less (tunc 1 vlna 1 fl – comparetur). Further on, in the solution, the sentence occurs, 'Nunc id est

* Numerus is denoted by o with a sloping line through it, radix by a sign resembling *r* with a loop attached, census by ʒ, cubus by a sign resembling *c* with a loop attached. For convenience of printing I replace the signs for numerus and radix by ϕ and ψ, and that for cubus by œ.

– 1 quam $\frac{60}{1\psi}$', *i.e.* 'Now it is less by 1 than $\frac{60}{1\psi}$'; and when it has been shown that in the first case 12 ells were bought, and in the second 15, the solution concludes 'Iam notas, quod 1 fl – vlna comparetur', *i.e.*, that an ell is bought for 1 florin less.

It may be noted that in this Latin Algebra a point is used as a square root sign, *e.g.*, .11 denotes $\sqrt{11}$.

The use of the signs + and – in the Latin Algebra will be further considered in §§ 166–167, 175–178.

§ 156. Thus Widman was in possession of two manuscripts, in one of which (the German Algebra) the sign – occurs, but apparently only in one paragraph, while in the other (the Latin Algebra) both – and + not only occur, but are used systematically.

Neither the German Algebra nor the Latin Algebra is in Widman's handwriting, but on the margin of the Latin Algebra at the side, and above and below the text, there are a number of problems and solutions which are in his writing. Four of these were printed by Wappler in his *Programm* (pp. 5–7), and thirty-eight more in the supplementary volume* to vol. xliv. of the *Zeitschrift* (pp. 539–554). In twenty-five of these solutions the signs + or – or both are used. Wappler also prints (p. 540) Widman's translation into Latin of a problem in the German Algebra in which he uses + and –.

Widman's University lectures, §§ 157–159.

§ 157. Notices announcing two courses of lectures on arithmetic by Widman at Leipzig occur in his own hand-

* This volume forms also the 9th volume of the *Abhandlungen zur Geschichte der Mathematik*. Wappler states that at first he was of opinion that only some of the problems added to the Latin Algebra were in Widman's writing, but afterwards he concluded that they all were written by Widman, though at different times (p. 541).

The volumes of the *Abhandlungen* originated (and were continued as far as the 10th volume) as supplements to the *Zeitschrift*. As references are sometimes made to the *Abhandlungen* as well as to the *Zeitschrift* the following table connecting the volumes of the *Abhandlungen* with those of the *Zeitschrift* will be found useful:

Abhandlungen, vol.	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
Zeitschrift, vol.	22,	24,	25,	27,	34,	37,	40,	42,	44,	45.

This table is derived from the "Generalregister zu Band 1–50 der Zeitschrift . . ." (Leipzig, 1905). After the 10th volume the *Abhandlungen* have double titles, but there seems to be nothing to show that the earlier volumes were also published apart from the *Zeitschrift*. The references in this paper will always be to the *Zeitschrift*, as being the more accessible and complete of the two periodicals; but confusion cannot be entirely avoided, as the supplements which have the *Abhandlungen* title-page may have been treated as separate publications and not bound up with the *Zeitschrift*.

writing on the fly-leaf of the volume C 80; and a notice of a course of lectures on algebra, also in his own writing, appears on the back of the leaf preceding the Latin Algebra. This last notice is very interesting, if only because it expresses Widman's great appreciation of the merits of algebra. It is addressed to students (Adolescentes Ingenui), and after referring to the common and rudimentary parts of arithmetic which have been most amply treated 'in our previous editions' (prioribus nostris editionibus), and which afford an easy method of calculation for general use, he proceeds "Yet if in human affairs anything more difficult or intricate should arise, it must be treated not by these, but by some higher, doctrines of numeration which Algobre, of most excellent and almost divine genius, has given to us in a few Aporismata (to use his own word), an art truly admirable and foremost of all the inventions of mortals, on account of its singular and concealed methods, and all the more so because, whether they have presented themselves with respect to numbers or matters connected with number, difficult and almost inexplicable problems, impossible for one ignorant of this art, can easily be investigated by the rules of this art, and since this matter conduces in the highest degree to the common utility of all, therefore to-day at the second hour after the discourse and the celebration of the bachelors' disputation Jo. W. De. Eg. will give exercises on the Aporismata and rules of Algebra at an hour and time to be agreed upon by the auditors".*

§ 158. We know that these lectures were actually delivered at Leipzig in the summer of 1486, for the students' notes (Kollegienheft) derived from them are preserved in Codex Lipsiensis 1470. These notes, which are entitled 'De regu-

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* "Si quid tamen in humanis negocijs ardius atque magis intricacius euenerit non illis sed altioribus quibusdam numerandi racionibus pertractandum erit quas preclarissimi quondam ac prope diuini ingenij Algobre paucis admodum Aporismatibus vt suo vocabulo vtar nobis tradidit artem sane admirandam ac inter cunctas mortalium inventiones precipuam tum propter singulares absconditosque calculandi modos. tum eo maxime quod siue de numeris siue de quibusvis rebus alijs ad numerum applicatis Enigmata difficillima ac pene inextricabilia apudque huius artis inscium impossibilia inciderit Artis huius Regulis facile investigari possint Que res cum ad communem omnium utilitatem summopere conducere videbatur Quare hodie hora secunda post sermonem atque Baccelaureorum celebrata disputatione Magister Jo. W. De. Eg. Aporismata et Regulas Algobre resumpturus pro hora atque loco conuenienti cum audeturis concordabit etc." (Wappler's *Programm*, p. 10). Wappler mentions that the same notice in a shortened form appears also on the fly-leaf of C 80. This notice Wappler subsequently printed in *Zeitschrift*, vol. xxxiv., supp., p. 167. It differs only in a few words, and is slightly shortened at the end. It will be noticed that Widman regarded Algobre as a person and Aporisma as his word. The word Aporisma is used in the Latin Algebra in C 80, and seems to mean a method of solution. I have not met with it in the other manuscripts or books that are referred to in this paper.

larum algebre conditione'*, occupy pp. 479–493′ of the volume, and at the end is the inscription "Hec Liptziensi in studio informata sunt a Magistro Johanne de Egra anno salutis millesimo 486 in estate in habitacione sua burse Drawpitz pro fl duobus, qui faciunt 42 gl argenteos". They were discovered by Curtze,† who found that they were identical with the Latin Algebra of C 80 published by Wappler in his *Programm*. He also states that in the college notes the signs + and − occur "in their present signification as in Widman's printed arithmetic". All this was confirmed by Wappler,‡ who considers that it is therefore proved that the Latin Algebra in C 80 was the foundation of Widman's lectures on algebra.

§ 159. Widman was inscribed a member of the University of Leipzig in 1480: he became Bachelor of Arts in 1482, and Master of Arts in December, 1485, or January, 1486§, and we have seen that he gave lectures on algebra in the latter year. Three years later he published his *Rechenung* (1489). Wimpina states that in 1498 he was living at his native place Eger, being about 31 years of age and continually doing new work‖ (Claret adhuc apud *Egrenses* annos natus uno forte supra triginta, continue nova cudens. A.D. 1498. sub *Maximiliano* Romanorum Rege). If Widman was about 31 in

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* The Latin Algebra in C 80 begins 'Pro regularum algabre cognicione est primo notandum.'

† "Eine Studienreise." (*Centralblatt für Bibliothekswesen*, Jahrgang 16, 1899, p. 289). Curtze in a note on p. 305 points out that 42 groschen was an enormous fee for the time, and quotes Günther with respect to other fees of the period, which vary from 2 groschen to 8 groschen for courses of lectures on Euclid, arithmetic, theory of the planets, etc. These lectures of Widman's are believed by Curtze to be the first public lectures on algebra that were given in a German university.

‡ *Zeitschrift*, vol. xlv, Hist.-litt. Abt., pp. 7–9. Wappler mentions that pp. 504–504′ of Lipsiensis 1470 also contain a portion of the Latin Algebra of C 80. He also found on p. 452 of this manuscript the following lecture announcement by Widman: "Concordia facta auditorum in 24 regulis algabre, et ea, que presupponuntur, puta, algorithmum in minucijs, in proporcionibus algorithmum, in additis et diminutis algorithmum, in surdis algorithmum, in applicatis, ceteros denique illis finitis algorithmos, vt in datis, de duplici differencia, in probis, non occultabit Magister Johannes de Egra, cras circa horam sextam et cetera post domici secunda feria". All these algorithmi are contained both in Lipsiensis 1470 and in Dresden C 80, so that Wappler concludes that Widman lectured on all these subjects as well as on the 24 rules of algebra.

The quotation in the text 'Hec Liptziensi...argenteos' differs slightly as given by Curtze and Wappler. I have followed the latter.

§ Drobisch, *De... Widmanni...compendio*, p 17. Boncompagni states (*Bullettino*, vol ix., p. 208) that Widman was admitted to the degree of Bachelor of Medicine in December 1485 and that of Master of Arts in January, 1486, his authority being this passage in Drobisch; but I think it is clear that the words 'Medic. Bacc.' in the passage do not refer to Widman.

‖ "Conradi Wimpinae...scriptorum insignium...centuria...luci publicae tradita a.. Merzdorf" (Leipzig, 1839), p. 50. Wimpina's notice of Widman is quoted in full by Boncompagni in vol. ix., p. 209, of his *Bullettino*.

1498 he was born about 1467, and therefore was only about 19 when he lectured on algebra in 1486. Wimpina gives the titles of four of his books (Algorithmi) besides his *Rechenung*. A complete list, containing six books* (besides the *Rechenung*), is given by Wappler in *Zeitschrift*, vol. xxxiv., supp., p. 167.

The signs + and – in the German and Latin Algebras in C 80, §§ 160–162.

§ 160. It would thus appear that the signs + and – became known to Widman by the German and Latin Algebras in C 80. It is only in the latter that the sign + occurs, and in it the signs + and – are used systematically, so that we may regard this Latin Algebra as the probable source of Widman's knowledge of them. Their use was extended by him both by his lectures and by his *Rechenung*, which is almost certainly the first printed book in which they occur.

§ 161. There seems to be nothing to suggest who was the author of either the German or the Latin Algebra in C 80. We know that the date of the former is 1481, and that the latter is not later than 1486. None of the earlier manuscripts that have been printed seem to contain any trace of the signs + and –, and it is possible that the unknown author of the Latin Algebra was the inventor of the sign + and the first user of + and – for the addition and subtraction of mathematical quantities. Wappler makes no suggestion with respect to the origin of this manuscript.†

§ 162. Of the two signs – is more freely used than + in the Latin Algebra, and it occurs also in the words of the text (*i.e.* not only connecting numbers and cossic signs), and it would seem not unlikely that + might have been introduced as a complement to – so that addition could be represented by a sign as well as subtraction. A sign for addition is not absolutely necessary, as the short word et could be used, but the use of a sign corresponding to – clearly renders the expressions more uniform in appearance and their treatment more symmetrical. Nothing throws any direct light on the origin of the signs

* Wappler considered that these were the books to which Widman referred in the words 'prioribus nostris editionibus' (§ 157).

† Eneström suggested that Widman was himself the author of the Latin Algebra, but merely on the ground that it was the basis of his lectures (*Bibliotheca Mathematica*, ser. 3, vol. iv., p. 90; vol. viii., p. 199).

themselves. It may be that − presented itself as the simplest abbreviation that could be found, and which had not already received a definite meaning. As for + it may have been derived from −, and distinguished from it by merely placing a vertical line across it, or it may have independently suggested itself as a variant of the abbreviation for et. It has a certain amount of resemblance to the mark used for et, and in a good many cases in the Latin Algebra it is used for et, and in several cases et is used where we should expect +. I infer from the Latin Algebra that + was intended to connect symbols or numbers by addition, and that where it connects ordinary words or sentences, or is used to introduce a consequence (as in + proueniunt, + manent, etc.), this is due to an error of transcription (see §§ 175–178).

Earlier manuscripts, §§ 163–164.

§ 163. Before making a more detailed examination of the Latin Algebra in C 80, it is convenient to refer to some of the earlier manuscripts on algebra in German libraries which have been printed, and to describe the means employed in them to denote addition and subtraction.

In vol. xl. of the *Zeitschrift*,* Curtze has printed ten extracts on the rule of false and on algebra from the volume no. 14908 in the Munich Library.

Of these extracts the first two (I and II) relate to the rule of false, and the others (III to X) to algebra.

In I and II (pp. 35–49), which are in Latin, plus and minus are applied to the error, but neither these words nor any other special words are used for addition or subtraction.

No. III (pp. 49–50) is a short algebra in German of about a page, which had been already printed by Gerhardt.† The word und is used for addition and minder for subtraction, *e.g.* (p. 50) "multiplicir die 2 dragmas minder ainer wurczen in sich selb, so komen 4 dragma vnd ain zins minder 4 wurczen" (multiply $2 - x$ by itself and there results $4 + x^2 - 4x$).

No. IV (pp. 50–58), which is entitled 'Regule delacose secundum 6 capitula', is in German. Subtraction is denoted by mynder (or minder), and minus is also used, as in the following sentence (p. 55): "Darumb multiplicir 1∂ wider den tayler, das ist 9 minus $\frac{1}{4}$ dings, so werden 120 mynder $2\frac{1}{2}$∂

* "Ein Beitrag zur Geschichte der Algebra in Deutschland im fünfzehnten Jahrhundert" (*Zeitschrift*, vol. xl., supp., pp. 31–74).

† *Monatsberichte* of the Berlin Academy for 1870, p. 142. Curtze here gives a more accurate version.

gleich. Nu multiplicir 1∂ wider 9 minus $\frac{1}{4}$∂, das macht 9∂ minus $\frac{1}{4}$ censo." [Multiply x by the denominator $9-\frac{1}{4}x$ and the product is equal to $120-2\frac{1}{2}x$. Now multiply x by $9-\frac{1}{4}x$, giving $9x-\frac{1}{4}x^2$].

Addition is denoted by und, mer, and juxtaposition, as (p. 56) 'multiplicir 20 fl uider 1 censy vnd 5∂, wirt gleich 200∂ vnd 500.' [Multiply 20 by x^2+5x and equate it to $200x+500$]; (p. 57) 'radix von $31\frac{1}{4}$ mer $2\frac{1}{2}$' [$\sqrt{31\frac{1}{4}}+2\frac{1}{2}$]; and 'Nu ist 3∂ minus 4 lb dem ander gleich, das ist 2∂ 10 mer' [$3x-4$ is equal to $2x+10$]; (p. 54) '1∂ $\frac{1}{20}$ censo' [$x+\frac{1}{20}x^2$]; (p. 56) '$\frac{100}{1∂5}$' $\left[\frac{100}{x+5}\right]$; (p. 57) '8 mol 1∂ vnd 4 macht 8∂ 32' [8 times $x+4$ is $8x+32$].

No. V (pp. 58–67), which is in Latin, contains the solutions of four problems, which are explained at great length. For addition the words et and cum are used, and for subtraction demptis or diminutis. Thus we have

$$\text{(p. 59)}\ \frac{32\text{ res et }45}{1\text{ census et }3\text{ res}};\quad \text{(p. 60)}\ \frac{15}{\frac{11}{14}\text{ et radix de }\frac{1381}{196}};$$

$$\text{(p. 63)}\ \frac{100}{1\text{ res demptis }5},\ \text{and}\ \frac{200\text{ res diminutis }500}{1\text{ census diminutis }5\text{ rebus}}.$$

The word minus is twice used for subtraction, viz. in '1 rem minus 5' and 'si minuo remanent $\frac{15}{2}$ minus radice de $\frac{125}{4}$; si addo, provenient $\frac{15}{2}$ et radix de $\frac{125}{4}$' (both on p. 63).

In no. VI (pp. 68–70), which is in Latin, minus is used for subtraction and et or juxtaposition for addition. Thus we have (p. 69) '1ψ minus 4 equatur $\frac{1}{2}\psi$ et 2', and '3ψ minus 3 | $\frac{1}{2}\psi$ $3\frac{1}{2}$' [$3x-3=\frac{1}{2}x+3\frac{1}{2}$].

No. VII (pp. 70–73) is partly in Latin and partly in German. For subtraction minus, minder, minner, or mynner is used, and the words for addition are mer and und, juxtaposition also being used: we have (p. 70) '$\frac{1}{3}$ von 6∂ minner 300 ist 2 ding minner 100' [$\frac{1}{3}$ of $6x-300$ is $2x-100$]; (p. 72) 'so pleibt 1∂ mer 27 gleich 2∂$\frac{23}{60}$' [$x+27=2\frac{23}{60}x$]; and 'multiplicir 15 mal 140.2∂, macht 2100.30∂...so pleibt 190∂ vnd 2 censy dem andern 2100 zall' [15 times $140+2x$ is $2100+30x$...there remains $190x+2x^2=2100$].

Nos. VIII and IX (p. 73) together occupy only 7 lines. Addition is denoted by juxtaposition and the word und.

No. X (p. 74) is in Latin, and the word minus is used. Addition is denoted by juxtaposition: thus (p. 74) '20 minus 1ψ in se sunt 1 census 400 minus 40ψ' [$20-x$ multiplied by itself is $x^2+400-40x$].

§164. In vol. xlv. of the *Zeitschrift*,* Wappler printed a lecture given at Erfurt by Magister Gottfried Wolack in 1467 and 1468. It is one of the manuscripts in C 80, and relates to the rule of three. No words are used for addition or subtraction, which, however, scarcely enter into the questions to which the lecture relates.

The German Algebra in C 80, §165.

§165. I now come to the German Algebra in C 80, which has been already referred to in §153. Wappler prints a portion of this Algebra on pp. 4–5 of his *Programm*, and an additional question from it in vol. xliv. of the *Zeitschrift*. In the part given in the *Programm* minner or mynner is used in subtraction, except in the passage where – occurs, and und is used for addition. The quotations in §§153–154 contain examples of the use of minner and und, and the passage in which – occurs. In the question printed in the *Zeitschrift* subtraction is denoted by mynner and addition by juxtaposition, *e.g.* $\overset{n}{140}\ 2^{c}$ is equivalent to $140n + 2c$. This manuscript is dated 1481.†

The use of the signs + and – in the Latin Algebra in C 80 *and by Widman*, §§166–169.

§166. The Latin Algebra (which is concerned only with the solution of simple and quadratic equations and problems giving rise to them) occupies 20 quarto pages (pp. 11–30) in Wappler's *Programm*. The primary rules (p. 11) are given in words, and neither – nor + is used; the word minus is not used to indicate subtraction, and plus does not occur at all. The 19 secondary rules are given symbolically, and in them + occurs no less than ten times (p. 13), always in expressions, such as 3ʒ + 4ꝑ. It first occurs in the text in the sentence 'multiplica 1ʒ et tres ϕ in se, veniet vnus ʒʒ + 6ʒ et 9ϕ equales 12ʒ' (p. 13). Here et is clearly right in the first expression, as it is followed by tres, but the second expression should be ʒʒ + 6ʒ + 9ϕ, and this is confirmed by the resulting equation, which is written 'vnus ʒʒ + 9ϕ equales 6ʒ'. As mentioned in §155, the sign – first occurs lower down on the same page in the sentence 'radix aggregati – medietate ꝑ est valor rei'.

* "Zur Geschichte der Mathematik im 15. Jahrhundert", *Zeitschrift*, vol. xlv., Hist.-litt. Abt., pp. 47–56. Wappler mentions that the manuscript has been corrected by Widman, who presumably used it in his teaching.

† *Zeitschrift*, vol. xliv., supp. vol., p. 539.

On p. 14 the sign + occurs once (in œ + ʒʒ), but et was probably intended. The first occurrence of the sign – in a formula is on p. 16 in the expression 10ϕ – 1ψ. In the subsequent pages + and – are freely used in formulæ.

Twice on p. 16, and a number of times on the next page, and occasionally on all the following pages, + occurs where it should be et. Thus + should be et in '+ radix' (p. 16); 'differencia...inter *d*+*c* 3 + inter *c* et *b* 4' (meaning 'differencia inter *d* et *c*, 3, et inter *c* et *b*, 4'), '+ proueniunt', and '+ denominator' (p. 17).

The words minus and plus (with the meanings 'diminished by' and 'increased by') are first used on p. 14, where the former occurs once and the latter three times, the following word in all four cases being medietate. The word minus in this sense occurs only twice more in the Algebra, viz. on p. 21, where it again precedes medietate, and on p. 24 where 10ϕ minus 1ψ is preceded and followed by similar expressions in which the sign – is used.*

After p. 14 the word plus is used about twenty times: it has the meaning 'increased by' in '+ plus dimidio' (p. 15); '1 et plus r tertij' (p. 16); and 'plus medietate' twice (p. 26): and it has the meaning 'more' in the other cases, as *e.g.* 'cum dimidio rei plus' (p. 15); 'in duplo plus te', 'plus quam in duobus' (p. 18); 'quod est 2 plus' (p. 27).

§ 167. A careful study of the Latin Algebra, as printed by Wappler, seems to me to show that + was a sign which was meant to be used in formulæ (*i.e.* in expressions connecting numbers and quantities involving the cossic signs) in the same manner as –, and as a complement to it. It is clear that in the manuscript of the Latin Algebra, as it appears in C 80, + must be quite distinct from the sign for et, or Wappler would not have printed + where the sense requires et: but in the original manuscript from which the Latin Algebra in C 80 was copied, the signs must have been sufficiently alike to cause the transcriber to make mistakes

On the wrapper of his *Programm* Wappler reproduces a portion of the Latin Algebra in facsimile. The sign for et occurs six times in this reproduced portion, but the sign + does not occur at all. The signs for et, which have some resemblance to a 7 with a horizontal line through it, differ somewhat among themselves. Another specimen of Widman's

* 'erit ergo maior pars 10ϕ – 1ψ. Multiplicando 10ϕ minus 1ψ per 1ψ proueniunt 10ψ – 1ʒ.'

handwriting is given by Wappler in vol. xxxiv., supp., p. 169, of the *Zeitschrift*. Here the sign for et occurs twice, but the sign + does not occur.

There is another manuscript in C 80 (pp. 295–300′) in which + and – are used. Extracts from it (for comparison with Riese's Coss) are given by Wappler on pp. 7–9 of his *Programm*. Wappler states (p. 7 of his *Programm*) that 35 of Riese's questions are taken from this manuscript, but he gives no further information about it, and he does not, I think, refer to it again. He does not seem to have directed his attention specially to the signs + and –, nor does he refer to their occurrence.

§ 168. As mentioned in § 156, Widman wrote a number of problems and solutions on the margin of the Latin Algebra, 42 of which have been printed by Wappler. In most of these solutions (as printed) the signs + and – are freely used for addition and subtraction in the formulæ, and there is no instance of + being used for et in the text. There is one problem, however, in which Widman has connected numbers by + instead of et. This is: Given two numbers 9 and 12, and another number 10, to find a fourth number 1ψ such that $9-\frac{1}{4}\psi$ is to $12-\frac{1}{4}\psi$ as 10 is to 1ψ. The statement of the question by Widman is as follows: "Propositis duobus numeris, scilicet $9+12$, si petitur ad quemlibet tercium, puta 10, aliquem numerum maiorem, cuius quidem maioris $\frac{1}{4}$ subtracta de primis duobus, scilicet $9+12$, residuum habeat, eandem proportionem quam numeri nunc ultimo inventi."* The value of 1ψ is found to be 16, and the solution concludes "cuius quartam partem, scilicet 4, aufero a $9+12$, et manent $5+8$ habentes eandem proportionem, quam $10+16$, quia ubique est dupla." Here + means et and not addition, but in the course of the work + is used correctly in $120+\frac{1}{4}ʒ$ and $480\phi+\frac{1}{1}ʒ$. I cannot explain this lapse except by supposing that Widman wrote + as he might have written the sign for et, or as a comma

* *Zeitschrift*, vol. xliv. (1899), supp. vol., p. 547. This curious question was probably suggested to Widman by a barter or exchange question which (in the manner in which it was solved) led to the proportion $9-\frac{1}{4}\psi : 12-\frac{1}{4}\psi :: 10 : \psi$. Widman's marginal question which I have quoted occurs on p. 357′ of the Latin Algebra (Wappler, *Zeitschrift*, *loc. cit.*, p 547) and the barter question on p. 358′ of the Latin Algebra (Wappler, *Programm*, p. 24); but it had been previously given in the extract no. IV (Regule delacose secundum 6 capitula) from the Munich Codex 14908 (Curtze, *Zeitschrift*, vol. xl., supp., p. 54). An erroneous solution of the barter question (which is itself confused) is there given, and in this solution, which is reproduced in the Latin Algebra, the value of ψ is found from the proportion $9-\frac{1}{4}\psi : 12-\frac{1}{4}\psi :: 10 : \psi$, which may have led Widman to form a question which should lead directly to this proportion.

might be written now, to connect, or separate, 9 and 12, and continued the use of the sign.*

§ 169. Coming now to Widman's *Rechenung*, the signs + and − are used as at the present time. When they are first introduced they are defined as denoting mer and minus (§ 11). In explaining the rule of false Widman uses the words plus and minus, but in the diagram they are replaced by + and −, which are placed between the position and error, as *e.g.* in

$$\begin{array}{c} 6 + \frac{7}{8} \\ \times \\ 7 + \frac{11}{16} \end{array}$$

Here the signs do not indicate addition or subtraction or any use of the word and.†

Eneström's views on Widman's use of + *and* −, §§ 170–172.

§ 170. In vol. ix. (1908-9) of ser. 3 of the *Bibliotheca Mathematica* Eneström‡ has maintained that in Widman + is not a true mathematical sign, and that it is only in certain places

* The same use of + for et occurs in a problem in Riese's *Coss*. This problem is one of those which are written on the margin of the Latin Algebra (§ 156) and which were translated into German by Riese. The question is to find two numbers in the proportion of 3 to 2, such that their sum is equal to their difference. The Latin original begins: "Dentur 2ϕ in proporcione sesquialtera", and + and − are not used in the solution. Riese's translation is "Item gib mir Zwu Zalnn die sich zusamen haltenn als 3 + 2 In proporcione sesqualtera so ich eyne Zal zur andernn addir Das gleich so nil komenn sam wan ich eyne mit der andernn diuidir thu im also setz die geringistenn Zalnn in proporcione sesqualtera wesende als 3 + 2 Multiplicir itzliche mit 1ψ werden $3\psi + 2\psi$ addir Zusamen komen 5ψ gleich so ich 3ψ in 2ψ diuidir als $1\frac{1}{2}\phi$ teyl ϕ in ψ kommen $\frac{3}{10}$ Multiplicir mit 3 + 2 werdenn $\frac{9}{10}$ vnnd $\frac{6}{10}$". Here + means and, and its use cannot be justified Riese's *Coss* was written in 1524, but was not printed. A full *résumé* of its contents, made from the original manuscript in the School Library at Marienberg, was published by Berlet in 1860 ("Programm der Progymnasial- und Realschulanstalt zu Annaberg") and again in 1892 ("Adam Riese, sein Leben . . . Die Coss von Adam Riese von...Bruno Berlet", Leipzig and Frankfort). Wappler has given the Latin text, and Riese's translation of this and other problems from the margin of the Latin Algebra on pp. 5–7 of his *Programm*. The problem is No. 38 of Riese's *Coss* (*Adam Riese* . . . p. 45). There are slight differences in the question as printed in Wappler's *Programm* (p. 5) and in Berlet's *Adam Riese* (p. 45). I have followed the former.

† Widman's *Rechenung* is the first printed book in which + and − are used in the rule of false, and it is not unlikely that he was the first person to so apply them.

Grammateus followed Widman in writing the position and error in the same line, but he omitted the cross, and wrote simply

$$\begin{array}{c} 300 + 77 \\ 240 + 77 \end{array}$$

This placing of the signs + and − between the position and error, *i.e.* between numbers not connected by addition and subtraction, persisted for many years (§§ 23, 85–87).

‡ pp. 155–158. See also vol. x., ser. 3, pp. 182–183.

that − is a true mathematical sign. In support of this opinion he says that in the first question (quoted in § 11), where + and − occur, the meaning of the words which define + and − is "the numbers which in the table stand after + denote overweight: the numbers which in the table stand after − denote underweight", and he then refers to 'darnach addir + vnd − zeusam' (§ 26), which he interprets as 'add the numbers which denote the surplus and the deficit' (addiere die Zahlen, die den Ueberschuss und das Fehlende bezeichnen). He says that the word mehr is only once identified with +, and that + means and, and is used in places where it cannot be read as plus, and he instances $\frac{3}{5} + \frac{2}{5}$* (§ 21), where these fractions are not to be added: and he expresses his conclusion in the words "Bei Widman ist das Zeichen + also nicht ein rein mathematisches Zeichen, und es ist darum irreleitend zu sagen, dass es als gewöhnliches Additionszeichen benutzt wird". He adds that it would be just as correct to say that every abbreviation used in the Middle Ages for et served as a usual addition sign. With respect to the sign −, Eneström says that it originally meant underweight or a small deduction, but that in some places it corresponded to our subtraction sign, and his conclusion "Bei Widman kommt also das Zeichen − zuweilen als gewöhnliches Subtraktionszeichen vor, aber es wird nicht regelmässig benutzt, und der Leser kann nicht einmal ersehen, wann und wie es zur Anwendung kommen soll".

§ 171. From what I have already said it will be seen that I disagree with Eneström, and think that Widman used + and − as algebraical signs in the same manner as they are used in the present day. Widman was an algebraist, for we know that three years before his *Rechenung* was published he lectured on algebra, taking the Latin Algebra in C 80 as the basis of his lectures; and it seems to me that in the *Rechenung* he merely extended to arithmetic the signs which he was familiar with in algebra, fully realising and appreciating the advantages of the use which he was making of them. As for the three cases where + occurs when it should have been und, viz. $\frac{1}{3}$ vn̄ $\frac{1}{4} + \frac{1}{5}$ (§ 9), 'Regula augmenti + decrementi' (§ 26) and '$\frac{3}{5} + \frac{2}{5}$' (§ 21), the first two are easily explainable as errors of the printer or author, and the third can be justified. The first expression occurs in the sentence "9 fl $\frac{1}{3}$ vn̄ $\frac{1}{4} + \frac{1}{5}$ eynss fl

* Eneström also refers to p. 131 of vol. i. of Tropfke's "Geschichte der elementar-mathematik" (1902) for other evidence. The only quotations on that page which seem to me appropriate are '$\frac{1}{3}$ vn̄ $\frac{1}{4} + \frac{1}{5}$' and 'Regula augmenti + decrementi' (§§ 9, 26).

wy kūmē 36 elln̄ machss alsso Addir $\frac{1}{3}$ vn̄ $\frac{1}{4}$ vn̄ $\frac{1}{5}$ zu sāmen kumpt $\frac{47}{60}$ eynss fl". It is clear that + is an error, due to the printer or writer, for they should be all vn̄'s or all +'s, and the + in 'augmenti + decrementi' is a similar error. As for $\frac{3}{5}+\frac{2}{5}$ the meaning is that if the mixture is to be worth 7 fl, $\frac{3}{5}$ consists of wine at 5 fl and $\frac{2}{5}$ of wine at 10 fl, and he verifies the result by saying that $\frac{3}{5}$ of 5 fl is 3 fl and $\frac{2}{5}$ of 10 fl is 4 fl, the sum of which is 7 fl.*

§ 172. I am surprised that Eneström should associate + and – with overweight and underweight in Widman merely because the first question in which they occur, and in connection with which they are explained, concerns weights. Widman's explanation is quite general, viz. that – is minus and + is mer, and he is just as ready to apply his signs to money as to weight. De Morgan and Gerhardt, who had only the *Rechenung* before them, might not unnaturally give a specific meaning to the signs on account of the kind of question in which they first occur (although I do not think that a study of the book itself supports this view), but Eneström knew of Widman's Latin Algebra in C 80 and of his marginal problems, and was aware that he was familiar with the use of the signs in algebra. Also in the question in which the words 'darnach addir + vnd – zeusam' occur Widman is following his own rule 'teyl. mit der minnerung vnd merung zusam geaddiret,'† and + and – stand for the merung and minnerung.

The sign – seems to me to be used by Widman exactly as it would be now, and on the whole to be used regularly, although it does not occur on all occasions when it could have been used.

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* A more doubtful use of + occurs in the next question, where wines worth 20₰, 15₰, 10₰, 8₰ are to be combined so that the mixture may be worth 12₰. Following the rule he has given, he doubles the value 12, making 24, from which he subtracts the sum of the two smaller values, viz. 18, leaving 6 'and so much take from that for 20₰ + 15₰' (so vil nym vonn dem pro 20₰ + 15₰). He then subtracts 24 from the sum of the larger values, 35, leaving 11, and 'so much must be taken from the two cheaper wines' (so vil muss ich nemē der geringern zweyer weyn). He then adds 11 and 11 and 6 and 6 and obtains the divisor 34, at which point he leaves the question directing the reader to follow the rule. If he had completed it in the same manner as in the previous question, corresponding to $\frac{3}{5}+\frac{2}{5}$ he would have written $\frac{11}{34}+\frac{11}{34}+\frac{6}{34}+\frac{6}{34}$, indicating that $\frac{11}{34}$ was to be taken of the 20₰ wine, $\frac{11}{34}$ of the 15₰ wine, $\frac{6}{34}$ of the 10₰ wine, and $\frac{6}{34}$ of the 8₰ wine. In 20₰ + 15₰ the + means and, for $\frac{6}{34}$ is to be taken from the 20₰ wine and from the 15₰ wine; but even here there is an underlying notion of addition. It is of course possible that the + in 20₰ + 15₰ is a misprint; as it also may be in $\frac{3}{5}+\frac{2}{5}$.

† The rule is quoted in the note on pp. 16, 17.

Irregularities of expression in Widman's Rechenung, §§ 173–174.

§ 173. Widman's book was a compilation from various sources, with probably some original questions. Presumably he frequently transferred questions just as he found them (or merely with the numbers altered), but sometimes he varied them, as in the eggs and pence question (see § 26), using + and −. The mode of construction of the book would thus suffice to explain any slight irregularities in the use of the signs. It is also to be noticed that Widman does not make any attempt at uniformity: *e.g.* in the same question he has 2 ct 18 lb and 3 ct + 5 lb; and he follows the loose phraseology of his time. Thus he divides 20 fl among several people so that the first has $1\frac{1}{2}$ fl + $\frac{1}{3}$, the second $2\frac{1}{2}$ fl + $\frac{1}{4}$, ...: he divides 100 fl among three people so that the first has $\frac{1}{3}-\frac{1}{4}$, the second $\frac{1}{4}-\frac{1}{5}$, the third $\frac{1}{5}-\frac{1}{6}$: a tile weighing 2 lb is broken into three parts of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$: three persons buy a horse to which they contribute respectively $\frac{1}{3}$, $\frac{1}{4}$, $\frac{2}{5}$.* In all these cases the numbers given merely indicate proportions. In one question he writes 'multiplicir $\frac{1}{3}$ $\frac{1}{4}$ $\frac{1}{8}$ durch eynãder facit 96',† but in a previous question where $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$ were involved, he had expressed himself more correctly, and directed that the denominators should be multiplied (multiplicir die nenner mitt eynander facit 105),‡ and he also does so later on when he has $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and directs that the denominators be multiplied, giving 24 and 60.§

§ 174. The question in which '$\frac{2}{3}$ vnnd 6 mer', '$\frac{3}{5}$ vnd 8 mer', etc., occur has been criticised in § 27, the problem being indeterminate as treated by Widman.‖ (See also §§ 51, 62, 186).

* It seems worth while to give the actual wording of the first three of these questions "Itm̄ 6 gesellen teylen 20 fl. Der erst sol habn̄ $1\frac{1}{2}$ fl + $\frac{1}{3}$ Der ander $2\frac{1}{2}$ fl + $\frac{1}{4}$ Vnd die anderñ 4 sollñ gleich teil haben. Nu ist die frag was ydem gepur zu seym teyl Machss also Reducir dye teyl facit $\frac{11}{6}$ + $\frac{11}{4}$ die sūmir fa. $\frac{110}{24}$ addir die 4 gesellen dar zu facit $\frac{206}{24}$ ist $\frac{103}{12}$ . . ." (p. 195′). Here the meaning is that the 6 portions are to be proportional to $\frac{11}{6}$, $\frac{11}{4}$, 1, 1, 1, 1.

"Itm̄ drey gesellen teylñ 100 fl vñ der erst sol habñ $\frac{1}{3}-\frac{1}{4}$ vnd der ander $\frac{1}{4}-\frac{1}{5}$ Vnd der drit $\frac{1}{5}-\frac{1}{6}$ Nu ist die frag &c. Machss also $\frac{1}{3}-\frac{1}{4}$ ist 12. Vnd $\frac{1}{4}-\frac{1}{5}$ ist 20. Vnd $\frac{1}{5}-\frac{1}{6}$ ist 30. Nu find eyn zal in der du haben magst $\frac{1}{12}$ $\frac{1}{20}$ $\frac{1}{30}$ Das ist 1800 . . ." (pp. 195′–196). Here it would be more correct to say that $\frac{1}{3}-\frac{1}{4}$ is $\frac{1}{12}$ &c., and to find a number which contains 12, 20, 30.

"Itm̄ es ist eyn Czigel der ist gebrochn̄ in 3 stuck das erst $\frac{1}{2}$ das ander $\frac{1}{3}$ dz drit $\frac{1}{4}$ vnd der czigel hat gancz gewegñ 2 lb Nu ist die frag wie vil itlichss stuck wigt . . ." (p. 146′).

† p. 149. ‡ p. 142. § p. 198.

‖ Eneström instances this question in support of his view that Widman treated mer as distinct from + ('kann nicht statt desselben + gesetz werden'). I think that $\frac{2}{3}$ vnd 6 mer, $\frac{2}{3}$ vnd 6, $\frac{2}{3}$ 6 mer, and $\frac{2}{3}+6$ would all have seemed equally correct and equally natural to Widman.

There is sometimes in Widman a certain balancing of the signs + and –: for example, in the question about the corals (§ 19), as the fractions of the weights are all positive, the weights are written $69\frac{1}{3}$, $59\frac{1}{4}$, etc., but in the money, as one amount contains a negative fraction, viz. $3-\frac{3}{4}$, the others are written $6+\frac{1}{8}$, $5+\frac{1}{8}$, etc.

The Latin Algebra in C 80, §§ 175–178.

§ 175. As has been said, there seems every reason to believe that Widman derived his use of the signs + and – from the Latin Algebra, and so far as I know this is the earliest manuscript in which they have been found to occur. It seems to me that the signs are used in this manuscript just as in Widman's *Rechenung*, and as they are used at present. Eneström, however, takes the same view of their occurrence in this manuscript as in Widman's book. He mentions* that et was used for addition in the Middle Ages (as *e.g.* in octo et tres), and it was natural that addition should be denoted by the same ligature without its becoming a mathematical sign, and he proceeds: "It is known that Widman had in his possession a manuscript Latin Algebra, in which + was used as a ligature for et when et was an addition-word, as well as otherwise". Now to me it seems clear that in the Latin Algebra + and – were purely mathematical signs, and that when + appears in the text with another meaning this is due to a slip in writing or an error of the transcriber.†

If + is merely a ligature for et, why does it not appear always for et? Why is it nearly always used in formulæ and only occasionally in the text, and when it does so occur why is it used in preference to et when it is surrounded by ets?

§ 176. In the Latin Algebra, as printed by Wappler in his *Programm*, the sign + does not occur on the first two pages (pp. 11 and 12): on p. 13 it occurs 13 times, always connecting two cossic terms: on p. 14 it occurs only once (in

* *Bibl. Math.*, ser. 3, vol. x., p. 182.

† I think there can be no doubt that + as a mathematical sign in the Latin Algebra represents et and not plus. In the Latin Algebra 'plus medietate', 'minus medietate' and '– medietate' occur, but '+ medietate' does not: if it had occurred it would have indicated that + stood for the word plus. In my view + was used to connect symbols, and when it occurs before a word it is a slip or a transcriber's error for et. There are two cases in which, standing between a symbol and a word, its meaning is doubtful, viz. 'habet...1ψ + duos ϕ' (p. 18), and '1ψ + tres ulne constant 60' (p. 22). I think that in both cases + is an error for et, but if + was used intentionally it would show conclusively that it stood for et and not for plus; for if it represented plus it would be followed by 'duobus ϕ' and 'tribus ulnis'. In the case of '+ plus dimidio' (p. 15) the plus may have been inserted by inadvertence, or it may have been intended for et, which latter supposition is supported by the fact that 'et plus r tertij' occurs on the next page.

œ + ʒʒ, where probably et was intended). On p. 15 there are 77 ets, and 3 +'s all of which mean et (regarding + in '+ plus dimidio' as meaning et): on p. 16 there are 48 ets, and 3 +'s which mean +, and 2 which mean et:* on p. 17 (on which occurs the greatest number of +'s for ets) there are 27 ets, and 19 +'s which mean +, and 8 which mean et: on p. 18 there are 38 ets, and 11 +'s which mean +, and 3 which mean et: on p. 19 there are 26 ets, and 11 +'s all of which mean +: on p. 20 there are 30 ets, and one + which means +, and 2 which mean et. In the next 10 pages there are 366 ets, and 49 +'s which mean +, and 13 which mean et.

Thus on pp. 16—30 (inclusive), which contain all the examples of the rules, there are 535 ets, and 122 +'s, of which 94 mean +, and 28 mean et: of course some ets may be errors for +'s, as in ʒʒ + 6ʒ et 9φ, referred to in § 166, but as et could always be used for +, it is not in general possible to determine which ets should be +'s,† although among the +'s the context almost always enables us to decide which should really be + and which should be et.

The principal uses of the word 'and' are (i) additive, (ii) connective, (iii) introducing a consequence, as *e.g.* in 'a hundred and three', '*A* and *B*', 'subtract *a* from *b* and there will remain', etc. In the whole Latin Algebra there are 139 +'s, of which 107 are additive, while 32 have non-additive meanings: and of these 14 are connective and 18 introduce a consequence.

§ 177. I do not think that I am straining what is likely to have happened in supposing that three +'s in Widman's *Rechenung* are slips or misprints, or that the +'s in the Latin Algebra, which do not occur in formulæ, are errors. There are certainly other printers' errors in the *Rechenung*, and Eneström has himself suggested that by a slip Widman has written Frontinus for Boethius‡; and, in regard to the Latin Algebra, he accepted Tropfke's suggestion that perhaps the sign used throughout for numerus should have been a variation of the initial letter of dragma, and that the unskilled transcriber wrongly copied it as the o with the sloping line through it.§

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* In '1ʒ et $\frac{1}{5}$ et 2 rebus + $\frac{2}{5}$' I regard the + as meaning et, as is shown by the context.

† Another example in which it is clear that et should be + is $\frac{480\psi \text{ et } 480\phi}{1ʒ + 2\psi}$, which is followed in the next line by $\frac{480\psi + 480\phi}{128ʒ + 256\psi}$ (pp. 26, 27).

‡ *Bibl. Math.*, ser. 3, vol. viii., p. 195.

§ *Id.*, p. 200. The Latin Algebra contains numerous errors and inaccuracies, many of which are probably due to the transcriber. Thus on p. 15, in the question in which 'cuius radix + plus dimidio' occurs, 4 is written for $4\frac{1}{6}$ and 2 for 7 in the final result, which is in effect given as $\sqrt{2\frac{13}{16}} + 4$ instead of $\sqrt{7\frac{13}{16}} + 4\frac{1}{6}$. On p. 16, in

§ 178. It is to be hoped that further investigation will show the origin of the Latin Algebra in C 80, and perhaps determine to whom we owe the signs + and –. The sign – was used in a hesitating and uncertain manner in the German Algebra, but both signs were freely used in the Latin Algebra.

All the evidence points to the signs having been introduced into mathematics by the German algebraists of the fifteenth century, and to Widman having transferred them from algebra to arithmetic, and it seems almost certain that he was the first to use the signs in print.

The word minus (or some equivalent) was a necessity in mathematics, but not so the word plus, for et could always be used in its place, except in the rule of false.

There seems to be absolutely nothing to show why – was used for minus. It was almost the simplest mark that could be made, and this may have been the origin of its use. It is likely that the sign + was suggested by a ligature for and, but it is not impossible that it entered algebra as a counterpart to –, distinguished from it by making a stroke through it.

There seems but little reason to connect + and – with any marks used in the warehouse.

In the case of printed books the sign – was already at hand among the printer's available type, but the sign + had to be specially made.*

Sources of Widman's Rechenung, §§ 179–186.

§ 179. Widman's *Rechenung* was derived from the Bamberg Arithmetic and from earlier mathematical manuscripts. The Bamberg Arithmetic has been fully described by Unger† and Cantor,‡ and Unger states that it was much used by Widman, who borrowed many passages word for word. The Bamberg Arithmetic was itself derived from manuscripts, one of which, the Algorismus Ratisponensis, has been described by Rath.§

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the question in which '1 et plus r tertij' $\left[1+\frac{1}{\sqrt{3}}\right]$ occurs, there is a confusion between the sixteenth and seventeenth rules, and the equation should have been given as '6 œ equantur 3ʒʒ + 2ʒ' instead of '6 œ et 3ʒʒ equantur 2ʒ'. On pp. 15 and 16 extrahere is written for subtrahere. On p. 13 the word census is defined, and on p. 22 the dative censui occurs, but on pp. 15, 16 censa, ʒ^{sa}, ʒʒ^{orum} occur as plurals; and there are many such irregularities.

* I have already remarked (p. 34, note) that it seems likely that the use of the words plus and minus (instead of + and –) by Grammateus in certain places was due to the want of type of a suitable size. See also p. 36, note.

† *Die Methodik*, pp 37–40.

‡ *Vorlesungen*, vol. ii. (2nd ed.), pp. 221–227.

§ *Bibl. Math.*, ser. 3, vol. xiii., pp. 17–22 ("Ueber ein deutsches Rechenbuch aus dem 15. Jahrhundert"). This manuscript is referred to in §§ 185–186.

This Algorithmus and other manuscripts, as well as the Bamberg Arithmetic, show that Widman was merely following the general practice of the time in giving a number of special rules, each relating to a particular and very restricted kind of problem. A considerable portion of the *Rechenung* must have been compiled by Widman from earlier writings on arithmetic; but some of the questions are of an algebraical character, and, as Widman was an algebraist, he would naturally have availed himself of the Latin Algebra in C 80 and other algebraical sources, and he may well have been the first to introduce some of these algebraical questions into arithmetic. It seems therefore to be of some interest to trace the history of a few of these questions as far as the writings already referred to in this paper* permit. Of course some of the algebraical questions could be solved by the rule of false, and so may be said to have already belonged to arithmetic; but those in which the solution depended on a quadratic equation could only have been derived from algebra.

§ 180. Under the heading 'Regula lucri'† Widman gave two examples which occur in the Latin Algebra in C 80. The problem to which the rule relates‡ is: Given the principal and compound interest for two years, find the interest for the first year; and the rule is: Multiply the principal by the compound interest, add to this product the square of the principal, extract the square root of this sum, and subtract from it the principal: *i.e* if a is the principal and b the compound interest for two years, then the interest for the first year is $\sqrt{(ab+a^2)}-a$.§

His examples are: (1) If in two years 20 fl have become 30 fl, what was the interest for the first year? The rule gives the interest in florins as $\sqrt{(20\times 10+400)}-20=\sqrt{600}-20$: and (2) If 25 fl have produced 24 fl interest in two years, what was the interest for the first year? The rule gives

$$\sqrt{(600+625)}-25=35-25=10 \text{ fl}$$

* viz. Wappler's *Programm* of 1887, and papers by him and Curtze in vols. xxxiv.–xlv. of the *Zeitschrift*.

† Widman, p. 127′.

‡ Widman gives rules without any indication of the kind of question to which they relate: this has to be inferred from the example or examples which follow. Thus in this case, after the heading Regula lucri, he proceeds: "Diese regel soltu alzo verfuren Multiplicir die hauptsum yn den gewin Darnach Multiplicir dy hauptsum in sich selbst quadrate Vnd addir das product zu dem ersten product Vnd die wurtzel der gantzen sum so du da von subtrahirest dy hauptsum. bericht den gewin der hauptsum Vnd Ist Recht."

§ Widman might have given the rule in the form "Add the principal to the compound interest and multiply the sum by the principal . . .", equivalent to $\sqrt{\{(a+b)a\}}-a$.

These questions are taken from the Latin Algebra.* The first question also occurs twice in the manuscripts printed by Curtze from the Munich Codex 14908 in vol. xl. of the *Zeitschrift*, and described in § 163. It first occurs in German in IV (Regule delacose secundum 6 capitula), p. 54, and in Latin in V, p. 61. In IV and in the Latin Algebra it occurs as an example of the 'Capitulum quartum', which relates to a quadratic of the form $x^2+ax=b$, giving the rule for its solution.

§ 181. Under the heading 'Regula Exessus'† Widman gives another rule which has been derived from a quadratic equation of the same form. The problem is: Given the product of two numbers, of which one exceeds the other by a given amount, to find the numbers, and the rule is: square half of the excess and add this square to the given product: extract the square root of the sum, and from it subtract half the excess: this gives the smaller number, *i.e.* if a is the excess and b the product, the equation is $x^2+ax=b$, whence $x=\sqrt{(\frac{1}{4}a^2+b)}-\frac{1}{2}a$. His example is "I have 4 florins more than you, and my money, multiplied by yours, is 96". The rule gives $\sqrt{(4+96)}-2=8$, which is the smaller of the two sums.

The problem described under the Regula excessus is to find two numbers when their difference and product are given. I do not find a question of this class in the Latin Algebra or elsewhere, which is curious, as a similar question in which the sum and product are given occurs in the Latin Algebra, viz. to divide 10 into two parts such that their product is 5.‡ The equation to which the latter question gives rise belongs to 'Capitulum quintum', which consists of the rule for the solution of an equation of the form $x^2+b=ax$.

§ 182. Cantor§, referring to the Regula lucri and the Regula excessus, asks whether Widman can really have failed to notice that he has taught the same procedure under two different names, and says that this would have seemed incredible, especially as Widman was acquainted with the 'rule

* Wappler's *Programm*, pp. 21 and 22.

† Widman, p. 118′ In the heading excessus is printed exessus. After the heading Widman proceeds: Also soltu procedirn in dieser Regl. Multiplicir der vbertretūg das halbe teyl ynn sich selbst vnd das product addir zu der hauptsum Darnach nym radicem quadratam des selbgñ aggregates vnnd do von subtrahir das halbe teyl der vntterscheyd ader vbertretung vnd das vberig ist die kleyner zal. zu welicher so du addirest die vbertretung erwechst auch die grosser".

‡ Wappler's *Programm*, p. 24.

§ *Vorlesungen*, vol. ii. (2nd ed.), p. 234.

called Algobre or Cosse', and had given lectures on algebra, and he adds that in spite of better knowledge he seems to have yielded to the tendency of the time to indulge in very many rules and conceal poverty of thought by abundance of names.

This criticism seems incomprehensible. Widman is giving rules for particular problems, and the rule or formula in the Regula lucri is not the same as in the Regula excessus, being $\sqrt{(ab+a^2)}-a$ in one case and $\sqrt{(\frac{1}{4}a^2+b)}-\frac{1}{2}a$ in the other. It is true that both rules are obtained by solving a quadratic equation of the same form, and if Widman had been writing an Algebra he would have put both questions under the 'capitulum quartum'; but in an arithmetic he could not have given the general rule for the solution of a quadratic equation of the form $x^2+ax=b$, nor indeed could he have shown that the problems in question depended upon such an equation.*

§ 183. The curious questions in which '6 eyer − 2ϑ' are bought for '4ϑ + 1 ey',† ϑ being the symbol for pence, would seem likely to have been contrived by Widman to display the use of + and −, and so it was taken to be by Cantor,‡ but in fact it is an old question which occurs twice in the first of the manuscripts described in § 163. The first of the two questions is "Septem ova demptis 2 denariis sunt empta pro 5ϑ et uno ovo: queritur quanti precii est ovum", and the second is "4 ova demptis 2 denariis emuntur pro 7ϑ et ovo, queritur quantum precium est".§ They are solved by the rule of false, and it is noticeable that demptis is used instead of minus, although plus and minus are used in the final statements of the positions and errors throughout this manuscript. It is difficult to see what would have suggested such a question originally. It afforded Widman a good example of the use of the signs + and −.

§ 184. Another kind of question of some interest is that which is given by Widman under the heading Regula augmenti + decrementi (§ 26). The first example is: if a man pays 12 pence a lb, he has 37 pence over; if he pays 15 pence

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* Eneström also has pointed out that Cantor's criticism is unjustified. *Bibl. Math.*, ser. 3, vol. viii., p. 195.

† Widman, p. 115.

‡ *Vorlesungen*, vol. ii. (2nd ed.), p. 230.

§ *Zeitschrift*, vol. xl., supp., p. 47. Eneström also has remarked that the question about the eggs and pence occurs in this manuscript (*Bibl. Math.*, ser. 3, vol. viii., p. 195).

a lb, he is short by 44 pence: how many lbs did he buy, and what was the amount of his money? In the second example, if a man pays his workmen 5 pence each, he has 11 pence over; if he pays 9 pence, he is 17 pence short. This question (with different numbers or in different forms) occurs in three of the manuscripts described in § 163. In manuscript II* it is solved by the rule of false, and in IV† and VI‡ by algebra. In II and VI it relates to the payment of workmen, and in IV to the purchase of ells of cloth. Widman's examples relate to the purchase of lbs of aniseed and the payment of workmen.§ His first question is noticeable because of his use of + and – to denote the terms to which these signs are prefixed (§§ 26, 172).

§ 185. Besides the questions depending upon quadratic equations and those involving + and – there are a number of problems and questions in Widman which are of some historical interest, as they have been taken from earlier writings. In the paper by Rath in the *Bibl. Math.*, referred to in § 179, he has described the Algorismus Ratisponensis (middle of the 15th century) and a Vienna manuscript Codex Vindob. 3029 (of about 1480): and he has compared them with the Bamberg Arithmetic (1483) and Widman's *Rechenung* (1489).‖ The third part of the Algorismus Ratisponensis relates to practical arithmetic, and Rath states that it contains about 250 problems, of which 43 are found in the Vienna manuscript, 42 in the Bamberg Arithmetic, and 51 in Widman, the questions having the same numbers or only differing slightly. He further states that 18 are contained in all three works, that 18 more are common to the Vienna manuscript and the Bamberg Arithmetic, while Widman has none in common with the Vienna manuscript alone, and only two with the Bamberg Arithmetic alone.

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* *Zeitschrift*, vol. xl., supp., p. 40. † *Id.*, p. 57. ‡ *Id.*, p. 68.

§ A similar question, relating to workmen, is given on p. 117 of Borgi (1484), and is solved by the rule of false. Rath states that questions of this type occur in the Algorismus Ratisponensis, the Bamberg Arithmetic, and the Stuttgart manuscript of 1488 referred to in the next note (*Bibl. Math.*, ser. 3, vol. xiv , p. 217).

‖ *Bibl. Math.*, ser. 3, vol. xiii., pp. 17–22. At the time of his death Curtze had prepared for press a copy of the Algorismus Ratisponensis, drawn up from the two Munich manuscripts Cod. lat. 14783 and 14908. This copy and Curtze's introduction have been used by Rath. Curtze states that both manuscripts of the Algorismus Ratisponensis emanated from the monastery of St. Emmeran at Regensburg, and that Frater Fridericus was the scribe and partly the author (1456–1461). Curtze also referred to the Algorismus Ratisponensis in the *Centralblatt für Bibliothekswesen*, Jahrgang 16 (1899), p. 286. Frater Fridericus was the scribe of some of the manuscripts referred to in § 163.

In a subsequent paper (*Bibl. Math.*, ser. 3, vol. xiv., pp. 244–248) Rath has given an account of another manuscript of date 1488 in the Stuttgart National Library.

Widman's book itself conveys the impression that several of his non-commercial questions (besides the familiar puzzle questions) were derived from previous writings without substantial alteration (see *e.g.* § 28); and Rath's comparison with the manuscripts and the Bamberg Arithmetic shows that this was the case.

§ 186. Rath mentions* that the Algorismus Ratisponensis has been used by succeeding writers in a 'somewhat uncritical manner' (in ziemlich kritikloser Weise), and he selects as an illustration the question which was quoted from Widman in § 27 (p. 18). Rath gives the problem and solution as follows: "384 fl are to be divided among 4 persons, so that A receives $\frac{2}{3}$ and 6 fl, B $\frac{3}{5}$ and 8 fl, C $\frac{5}{6}$ and 10 fl, D $\frac{7}{8}$ and 6 fl. Find the chief denominator 720, the half of it is 360; $\frac{2}{3}$ of this is 240, 6 added gives 246. So many parts belong to A; for B, C, D we find 224, 310, 321 parts. In these proportions the sum is to be divided".† Rath states that this solution occurs in almost the same words (diese Lösung findet sich in fast wörtlicher Uebereinstimmung) in the Algorismus Ratisponensis, the Vienna manuscript, the Bamberg Arithmetic, and in Widman. It will be noticed that 720 is the number obtained by multiplying together all the denominators, and that it is halved, giving 360, the number used: but there is no apparent reason why it should not have been divided by 3 or 6, giving 240 or 120, both of which numbers contain all the denominators. Widman at once takes the number 360 (Find eyn zal dar yn du die gebrochen alle habst Vn̄ ist 360‡) without the intervention of 720.

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* *Bibl. Math.*, ser. 3, vol. xiii., p. 22.

† "384 fl. sind unter 4 Personen so zu teilen, dass A $\frac{2}{3}$ und 6 fl., B $\frac{3}{5}$ und 8 fl., C $\frac{5}{6}$ und 10 fl., D $\frac{7}{8}$ und 6 fl. erhält. Man sucht den Hauptnenner 720, die Hälfte davon ist 360; $\frac{2}{3}$ hiervon ist 240, 6 addiert gibt 246. Soviel Teile bekommt A; für B, C, D findet man 224, 310, 321 Teile. Nach diesen Veshältniszahlen wird dann die Teilung der Summe vorgenommen."

‡ Huswirt (*Enchiridion*, 1501) gives this question (on D iii) with the same numbers as in the Alg. Rat. and in Widman, and he follows the former in first taking the number 720 and then halving it (q̄re numerum...in quo oēs hos denoītores habere possis. et est 720 q̄ media. et relictū erit nūer⁹ q̄sit⁹ scz 360).

A similar question in which the fractions are the same is given by Tonstall ("De arte svppvtandi libri qvattvor", London, 1522), but he takes the least common multiple of the denominators, viz. 120. This question (on C 4) is to divide the sum of 600 aurei among four persons so that they have $\frac{2}{3}$ and 9, $\frac{3}{5}$ and 8, $\frac{5}{6}$ and 7, $\frac{7}{8}$ and 6. His direction is "In primis minimus numerus, qui omnes has denominationes capiat: inquirendus est. Is autem est centū et uiginti."

Questions of this type have been referred to in this paper in § 27, p. 19 (text and note), § 51, p. 38 (text and note), § 62, p. 45. In the question from the *Lilium* referred to in the note on p. 19 the product of the denominators 2, 3, 5, viz. 30 is also the least common multiple. In the other questions in Paciolo, Rudolff, and Riese a better mode of solution is adopted.

Widman occasionally gives trivial or ambiguous questions, but he does not indulge to any considerable extent in the enigmata or puzzle questions which were popular at the time and afterwards. One question of this kind which may be noticed is that of the three women with 10, 30, and 50 eggs, who sell them at the same price and bring back the same money. This is an old question which occurs in the early manuscripts, and, although Widman gives it in a confused form, he seems to have made an independent examination of the arithmetical principles on which it depends.*

The headings in Widman's Rechenung, § 187.

§ 187. Widman often gives a heading to a rule, as in the case of the Regula lucri and Regula excessus already mentioned† (§§ 180–182). The rule itself is followed by one or more examples. Frequently the heading has no obvious connection with the subject of the rule, which is itself without meaning, until an example shows the kind of problem to which it is to be applied. Widman also very often supplies headings to the separate examples or questions, which consist of the name of the article to which the question relates, such as figs, soap, aniseed, cloves, etc., or of some word or words which occur in the statement of the question, such as 'Schuch', 'Hering', 'Wol Tuch', 'Schoff Esel Ochsen', etc.‡ It seems to me that Widman's intention in giving names as headings to the rules, and headings to the examples and questions, was merely to break up the text so that the eye could more readily separate the different rules, examples, and problems. This view is confirmed by such headings as Regula pulchra, Regula plurima, Regula bona, and we may suppose that in these cases Widman was unable to find a more suitable title.

The questions which were to receive names seem to have been rather arbitrarily selected, and certain types for which a name would have been convenient have not been provided with a heading; for example, there is no name for the class which may be represented by the question of the cask with three taps, the times in which each separately could empty it

* I defer the discussion of this question and of Widman's treatment of it to another paper.

† These Regulæ are: Regula inventionis, fusti, transversa, ligar, positionis, equalitatis, legis, augmenti, augmenti + decrementi, sententiarum, suppositionis, residui, excessus, collectionis, quadrata, cubica, reciprocationis, lucri, pagamenti, alligationis, falsi; also regula pulchra, which occurs five times, regula plurima, and regula bona

‡ Including the regulæ, but excluding the heading Proba, there are about 140 headings.

being given, and the time in which all three running together could empty it being required. Other questions of this class relate to three mills grinding corn; a lion, wolf, and dog eating a sheep; and a ship with three sails. These four questions are placed together in Widman (pp. 136—138′), but there is no heading for the class, though each question has its separate heading, 'Eyn fasz mit dreyen Czapfen', 'Von der Mulen', 'Leb, wolff, hunt',* 'Schiff'. It may be noticed that Widman does not use the title Regula virginum or Regula cœcis, although he gives a question which belongs to this title (p. 160′). There is nothing to indicate which, if any, of the names of rules were of Widman's invention. This may have been the case with at least some of them, for Cantor and Unger mention only headings of chapters in the Bamberg Arithmetic: and there are no special names in Borgi's *Opera de Arithmethica* of 1484, which is the only arithmetic of earlier date than Widman's that I have seen. In the *Liber Abbaci* of Leonardo Pisano the numerous headings were generally suggested by the concrete form in which each special question or class of questions was presented as, *e.g.* 'de inventione bursarum', 'de emptione equorum inter consocios', &c. If there are headings in the manuscripts described by Rath† and in the Bamberg Arithmetic, it would be interesting to compare them with those of Leonardo and Widman.

Not many of Widman's names were adopted by his successors, though of course such names as Regula de Tri, Regula alligationis, Regula falsi necessarily occur in arithmetics where these subjects are included.

The word minus in the Bamberg Arithmetic (1483). *Widman not the originator of questions having minus in the data*, §§ 188—194.

§ 188. It was not until after the completion of this paper and when nearly the whole of it had been published that I received the second edition of the first portion of Tropfke's *Geschichte der elementar-mathematik*‡, published in 1921. This portion, which contains 177 pages, corresponds to the 'Erster Teil. Das Rechnen' of vol. i. of the first edition

* The 'lion, wolf, dog' question is quoted by Unger (*Die Methodik*, p. 41), who however says it occurs under Regula lucri.

† *Bibl. Math.*, ser. 3, vol. xiii., pp. 17—22; vol. xiv., pp. 244—218.

‡ "Geschichte der elementar-mathematik in systematischer darstellung mit besonderer berücksichtigung der fachwöter von Dr. Johannes Tropfke direktor der Kirschner-oberrealschule zu Berlin erster band rechnen zweite, verbesserte und sehr vermehrte auflage" (Berlin and Leipzig, 1921, pp. viii + 177).

(1902), which contains 120 pages. The quotations in this edition show that at least two questions in Widman in which – occurs were taken from the Bamberg Arithmetic, the word minus being replaced by the sign.

§ 189. Under 'Die Tararechnung' on p. 168 Tropfke, after stating that the Arabic word tara and the Italian words brutto and netto were not used in German mercantile computations in the 15th century, proceeds "Das Bamberger Rechenbuch (1483) enthält Angaben wie: *Itē 1 sack piper wigt* $2\frac{1}{2}$ *ct min⁹ 9 lb' vnd kost ye 1 lb'* 8 *ss minus* 3 *hell' vn̄ sol fur den sag abschlahē* 3 *lb'* $\frac{3}{4}$ *was kost das alles.* Das Wort *minus* ist hier also keinesfalls für unser Tara zu nehmen", and he adds in a note 'Gegen Cantor, *Vorlesungen* 2², S. 224'. He then proceeds "Minderwertige, unreine Ware, Rückstand usw. wird als *fusti* (italienisch = Stengel) bezeichnet, so bei Nelkengewürz *negelin*, bei denen in einer Aufgabe *ye* 1 *ct* 13 *lb' fusti* enthält; während hier das Pfund *lauter negellē* (reines Nelkengewürz) mit 13 Schilling 3 Heller berechnet wird, wird die *fusti* mit 2 Schilling minus 3 Heller angesetzt. Eine Erklärung des Wortes *fusti* wird nicht gegeben. Widmann behandelt solche Aufgaben in seinem Rechenbuch von 1489 in einer besonderen *regula fusti*". Then follows the extract from Koebel which I have quoted in § 106 (p. 74).

§ 190. The first of the questions quoted by Tropfke from the Bamberg Arithmetic is the same as Widman's pepper question quoted in § 16 (p. 10); and the second is probably the same as Widman's first question under Regula fusti quoted in § 89 (p. 65), for 'ye 1 ct 13 lb fusti' is the same, 2 ss minus 3 hlr is the same, and the only difference is in the cost of the pure cloves which in Widman is 11 ss 3 hlr and in the Bamberg Arithmetic (as quoted by Tropfke) is 13 ss 3 hlr.

Thus there are at least two questions in the Bamberg Arithmetic in which minus is used in the data, money being so expressed in both, and weight also in one; and these questions were transferred by Widman to his *Rechenung*, the only changes being that the word minus was replaced by the sign –, and that he wrote 3 lb + $\frac{3}{4}$ instead of 3 lb $\frac{3}{4}$ (though in the solution it is written 3 lb $\frac{3}{4}$). In stating the deduction to be made for the weight of the sack Widman follows the Bamberg Arithmetic in the words 'sol fur den sack abschlahn̄'.

§ 191. In § 39 (p. 27) I said that from the accounts given by Unger and Cantor of the Bamberg Arithmetic it did not

seem likely that it contained any question in which minus or its equivalent occurred in the data, and that as such questions did not occur in the early Italian Arithmetics, it seemed possible that such questions might have originated in Widman's desire to exhibit the uses of the signs + and −.*

This suggestion is completely negatived by the two questions quoted by Tropfke, which show conclusively that Widman was not the originator of such questions, and that he transferred at least two of them from a preceding work, merely replacing the word minus by the sign −.

§ 192. Unger's account of the Bamberg Arithmetic is given on pp. 37–40 of his *Die Methodik*, where he states the nature of the contents of each of the 21 chapters. Cantor, who mentions that through the kindness of Dr. Unger he has received from him a transcript of the whole book†, also gives an account of each of the chapters, and in describing chapter 10, on the rule of three, he writes, "f. Anwendung der Regeldetri in Waareneinkaufsrechnungen. Was wegen Verpackung nicht als Waarengewicht mitzurechnen ist und später Tara genannt wurde, heisst hier einfach *das Minus* und wird subtrahirt".‡

In the examples quoted by Tropfke minus is used in the sense of 'diminished by', and the words 'fur den sag abschlahē 3 lb' $\frac{3}{4}$' show that in this case the weight of the receptacle is not called 'das minus', and Tropfke's note 'Gegen Cantor, *Vorlesungen* 2^2, S. 224' seems to imply that Cantor was in error in his general statement on this point. In § 108 (p. 77) I wrote (with reference to Cantor's statement) "Without examining the book itself it is not possible to judge whether 'das minus' was used as a special term for tara, or merely meant that it was a minus quantity and therefore to be subtracted. The latter view would seem the more probable". It now seems doubtful whether 'das minus' is used in any question for the tara,

* "I have not seen the Bamberg Arithmetic (1483), but from the accounts of it given by Unger and Cantor it does not seem likely that it contains any question of this type (*i.e.* in which some of the data are expressed as one amount diminished by another). I have found no such question in Borgi (1484) or Paciolo (1494), so that it is quite possible that this kind of question originated in Widman's desire to exhibit the uses of + and −" (§ 39, p. 27).

† On the book itself Unger has the note "Exemplar in Zwickau, Rathsschulbibliothek; wahrscheinlich Unicum" (*Die Methodik*, p. 37): but in a note on p. 55 of the new (1921) edition of his *Geschichte*, Tropfke states that a second copy has been found by H. Wieleitner in the Augsburg Town Library.

‡ *Vorlesungen*, vol. ii. (2nd ed.), p. 224. Cantor repeats this statement on p. 231. "Nun ist ja richtig, dass im Bamberger Rechenbuche (S. 224) das Bruttogewicht zum Nettogewichte gemacht wird, indem man die Verpackung als 'das Minus' abzieht."

and it is very desirable that this point should be settled. If 'das minus' is not so used, it would be interesting to know whether minus occurs at all in connection with a deduction for the weight of the receptacle or packing.

§ 193. Tropfke's quotations from the Bamberg Arithmetic throw further discredit on the view that the signs + and − came from the warehouse; for, wherever these questions came from, they did not bring with them the signs. It seems fairly clear that the signs belonged to algebra, and that Widman in writing his *Rechenung* merely applied them to existing examples which he found suitable.

§ 194. Questions in which minus or its equivalent is used in the data do not seem to occur in the early Italian Arithmetics, so that they were probably of German origin. Widman followed the Bamberg Arithmetic in giving questions of this kind, and he was followed in this respect by his successors, as was shown in §§ 43–63. It would be very interesting to know if any other questions of Widman's in which the sign − occurs were taken from the Bamberg Arithmetic, and especially interesting to know whether the Bamberg Arithmetic contains an example of the same kind as Widman's fig question, or one which may have suggested it to him.

Another matter of great interest would be to trace to their original sources questions in which minus occurs in the data by examining with this object the Algorismus Ratisponensis and other manuscripts from which the writer of the Bamberg Arithmetic may have derived them.

The use of the symbols × *and* = *in addition, subtraction, multiplication, division, and the rule of three. De Morgan's criticism*, §§ 195–200.

§ 195. In the paper in the *Camb. Phil. Trans.*, which has been referred to several times (pp. 1, 9, 38, &c.), De Morgan observes that the invention of signs of operation did not commend itself to the arithmeticians of Widman's time, and he comments on the slowness of growth of symbolic language which characterised the period 1450–1550.* The case he takes is that of the guide lines used to indicate the process of finding a fourth proportional to three given fractions as in

$$\frac{2}{3} \times \frac{4}{5} \frac{-}{-} \frac{6}{7},\dagger$$

* *Camb. Phil. Trans.*, vol. xi., pp. 205–206.

† De Morgan's example is $\frac{32}{1} \times \frac{69}{4} \frac{-}{-} \frac{3}{1}$, which occurs on p 78' of Widman, and is his first use of × and =.

where the lines show that a fourth proportional to $\frac{2}{3}, \frac{4}{5}, \frac{6}{7}$ is obtained by multiplying 3, 4, 6 and 2, 5, 7. He states that the "arithmeticians' own *directive* symbols never became signs of *operation*, though in almost universal use for a century": and he considers that × and = should have become separate symbols denoting division and multiplication, *i.e.* that $\frac{2}{3} \times \frac{4}{5}$ should have meant $\frac{4}{5}$ divided by $\frac{2}{3}$, and $\frac{4}{5} \frac{-}{-} \frac{6}{7}$ should have meant the product of $\frac{4}{5}$ and $\frac{6}{7}$; and that in general × placed between two quantities should have denoted that the second should be divided by the first, and = placed between two quantities should denote that they were to be multiplied together.

§ 196. In the course of the preparation of this paper I examined the uses of crossed and other indicating lines in the different books which I consulted, and I had intended to include these results (like those relating to tara and fusti) in the present paper: but this intention was abandoned on account of the additional space that would be required. I will here only say that I did not find myself in agreement with De Morgan. The guide lines were not so generally employed in the rule of three as his statements, and the list of authors he gives, would imply; and he ignored the fact that the crossed lines had other uses. The cross was placed between two fractions, as in $\frac{2}{3} \times \frac{4}{5}$, not only when they were the first two terms in the rule of three (so that $\frac{4}{5}$ was to be divided by $\frac{2}{3}$) and in division, but also when the two fractions were to be added or one was to be subtracted from the other*; and there are cases in which $\frac{2}{3} \times \frac{4}{5}$ meant that $\frac{2}{3}$ was to be divided by $\frac{4}{5}$: in fact the cross formed of two guide lines had a much wider meaning than a direction to divide the second fraction by the first, and it would not have been natural to restrict it to this purpose. Also, the guide lines being so closely associated with the multiplication of the numbers which they connected, the transition from $\frac{a}{b} \times \frac{c}{d}$ to $a \times c$ would not be so easy, as unit denominators would have to be understood, *i.e.* $\frac{a}{1} \times \frac{c}{1}$. It is a long step to pass from a cross which, placed between two *fractions*, merely indicated that among the operations to be performed the numerator of

* In the case of addition or subtraction a line was often placed beneath the cross to indicate that the denominators were also to be multiplied together.

each was to be multiplied by the denominator of the other, to a cross placed between two *numbers* which was to indicate that the second was to be divided by the first. Even at the present time there is no recognised sign which, placed after a and before b, denotes the fraction $\frac{b}{a}$, although $a \div b$ and a/b are frequently used to denote the fraction $\frac{a}{b}$.

Also, as in the fraction $\frac{a}{b} \frac{-}{-} \frac{c}{d}$, a line indicates that a and c are to be multiplied together, it would seem natural to use $a - c$ to indicate that a and c are to be multiplied together when a and c are any numbers as well as when they are both numerators or both denominators; but, this being impracticable owing to the other uses of $-$, I do not think that it would have seemed natural to replace $a - b$ by $a = c$, for if in $\frac{a}{1} \frac{-}{-} \frac{c}{1}$ the denominators were omitted so also would be the line connecting them.

§ 197. In some works the cross was used in addition, subtraction, and division of fractions, but not in the rule of three: in others it was used in the rule of three only, and sometimes it was used in all these processes. In division the divisor was generally placed first, but Gielis vander Hoecke in his *Arithmetica** placed the dividend first, as in '$\frac{3}{5} \times \frac{3}{7}\ \frac{21}{15}$ facit $1\frac{2}{5}$', although he uses $\times$ in the rule of three as in

$$\text{‘}\frac{4}{5} \times \frac{5}{6} \frac{-}{-} \frac{7}{9} \text{ facit } \tfrac{175}{216}\text{’}.$$

Apianus in his *Rechnung* of 1527† separated the terms of a proportion by lines as in '$\frac{7}{2} - \frac{2}{1} - \frac{3}{4}$', and his direction to obtain the fourth term is 'multiply the middle and last terms and divide by the first', which in this case he expresses by '$\frac{2}{1} \frac{-}{-} \frac{3}{4}$ ft. $\frac{3}{2}$ tayl in die erst $\frac{3}{2} \times \frac{7}{2}$ ft. $\frac{3}{7}$ fl', so here the dividend is to the left of the cross. Similarly in the case of the proportion '$\frac{100}{1} - \frac{3271}{2} - \frac{32}{3}$' the working is '$\frac{104672}{6} \times \frac{100}{1}$ ft.

* I quote from the Antwerp edition of 1544, the title of which begins "In Arithmetica, een sonderlinge excellēt boech leerende . . . ." The quotations are from pp. 29′ and 32′. There was an earlier edition of 1537, of which the title page is reproduced in *Rara Arithmetica*, p. 184. Vander Hoecke uses $\times$ in the addition, subtraction, and division of fractions and $=$ in the multiplication and in the rule of three. He does not insert the horizontal line below the cross in addition and subtraction.

† The title is given on p. 32. The quotations are from F viii′, G i, E viii′ and F i.

fl $174\frac{34}{75}$'. But he then points out that the better method is to take the numerator of the third term as a new third term, to multiply together the denominators of the second and third terms and the numerator of the first for a new first term, and to multiply the denominator of the first term and the numerator of the second term for a new second term, thus obtaining three terms which are free from fractions. This he illustrates by the diagram

$$\tfrac{7}{2} \times \tfrac{2}{1} - \tfrac{3}{4},$$

where the guide lines indicate the formation of the three terms 28, 4, 3.

Though Apianus did not actually use the cross in addition, subtraction, or division of fractions, he symbolised the three processes by $\underline{\mathbf{x}}$, $\underline{\mathbf{x}}$, $\mathbf{x}\times^{\circ}_{\circ}$ and multiplication by $=$. He uses the double cross for division because he places the dividend first and has to invert the resulting fraction. This appears from his rule which is 'Setz den Tailer zü der rechten handt, vnd multiplicir durch kreutz', and from his example '$\frac{1}{2}$ Tail in $\frac{2}{3}$ facit $\frac{3}{4}$'. Thus both of these writers were irregular in their use of the cross, and it is clear that they regarded it as merely indicating the pairs of numbers which were to be multiplied. Tonstall in his *De arte supputandi* (London, 1522) lays stress on the fact that it is immaterial on which side of the cross the divisor is placed, though he himself prefers to place it first, as he does so in the division of numbers.*

Cardano† uses the cross in addition and subtraction of fractions, the arms of the cross as usual indicating the numbers which are to be multiplied together: but in the rule of false the arms indicate subtraction or addition. He places the first error under the first position, and the second error in line with the first position and over the second position, and connects the positions and the errors by a cross: thus the two errors are at the ends of an arm, which indicates that the smaller is to be subtracted from the larger (or added if of opposite signs) and similarly for the errors: these differences (or sums) are the first two terms of a proportion of which the error is the third term, the result being the correction to be applied to the corresponding position.

* His words (on S 4') are "Sunt: qui iubent in minutijs diuidendis diuisorem à dextra poni: quasi id magni referat qui cur id fieri sic uelint: nihil uideo: quando illi ipsi præcipiunt fragmenti diuidendi numeratorem in diuisoris denominatorem, contraque diuidendi denominatorem in numeratorem diuisoris duci. Quod præceptum si quis seruet: nihil omnino refert: ab utra parte diuisor steterit . . ."

† "Hieronimi C. Cardani...practica arithmetice..." (Milan, 1539). The rule of false is explained in chapter xlvii.

§ 198. De Morgan stated that the "lines guided the details of these operations [division and multiplication], but never symbolised the operations themselves".* The more correct statement, it seems to me, would be that in the treatment of fractions a single line 'guided the details' of the operation of multiplication and in fact symbolised the operation itself, and for this reason the crossed lines and parallel lines each symbolised two multiplications. Thus $\frac{a}{b} \times \frac{c}{d}$ denoted the formation of *ad*, *bc* without regard to whether they were to be combined in $\frac{ad}{bc}$, or $\frac{bc}{ad}$, or $\frac{ad+bc}{bd}$, or $\frac{ad-bc}{bd}$. Since $a-c$, occurring in a fraction, symbolises the product *ac*, therefore $\frac{a}{b} \frac{-}{-} \frac{c}{d}$ symbolises $\frac{ac}{bd}$. It is clear that at a time when $\times$ and $=$ were used in connection with fractions the extension of these marks to indicate processes applied to simple numbers would have been difficult; and as we have seen even in fractions the cross was used in addition and subtraction as well as in division, so that it would have been impracticable to apply it to division only.

It may be mentioned that though Tartaglia† uses $\times$ and $=$ in the rule of three as in $\frac{17}{3} \times \frac{31}{3} \frac{-}{-} \frac{108}{8}$, these fractions being the first three terms of a proportion, he writes, in division of fractions, 'a partir per $\frac{2}{3} \times \frac{3}{4} \frac{-}{-} \frac{9}{8}$', where the horizontal lines merely connect the products with the factors.‡

Although single lines were sometimes used to separate the terms in a proportion, and to connect or separate numbers in other ways, their most usual purpose was to indicate multiplication, and it seems quite possible that – might have

* *Loc. cit.*, p. 206.

† "La prima parte del general trattato di nvmeri . . ." (*Venice*, 1556). The quotations are from pp. 145′, 117.

‡ In division of fractions the rule 'Invert the divisor and proceed as in multiplication' is, I think, due to Stifel, who gives it in his *Arithmetica Integra* (1544) and seems to claim it as his own. Under 'De Divisione Minutiarum' he writes "Ego Diuisionis regulam reduco ad regulam Multiplicationis Minutiarum, hoc modo: Diuisoris terminos commuto, id est, numeratorem pono sub uirgula, & denominatorem supra uirgulam pono. Hoc facto, nihil aliud restat, nisi ut opereris iuxtā regulam Multiplicationis superius datam" (p. 6). He gives this rule also in his *Deutsche Arithmetica* (1545); and, in his edition of Rudolff's *Coss* (1553), after saying that Rudolff's rule for division [to reduce the fractions to a common denominator and then divide one numerator by the other] is 'wol. künstlich gemacht,' he expresses his preference for his own rule "Aber doch ist meyn Regel vom diuidiren der brüch, viel gebreuchlicher (wie mich bedünckt) deñ des Christoffs": and then he explains the rule (*Die Coss*, p. 25). It is to be remembered, however, that Stifel used 'dieser meiner Zeichen' in a case where I do not think he meant to claim the signs as his own. See § 59 (p. 42).

become the sign for multiplication (had the need of such a sign been felt) if it had not been already appropriated for subtraction.

§ 199. Widman's *Rechenung* was the earliest book in which De Morgan had met with the guide lines $\times$ and $=$ in the rule of three. Widman uses them five times, but he does not use the cross in addition, subtraction, or division, nor the parallel lines in multiplication: he regards them however as indicating division and multiplication, for, after his first use of them to connect the three terms of a proportion, he gives the direction 'multiply the middle by the last and divide the product by the first'.

Borgi (1484) used the guide lines not only in the rule of three, but also in the four fundamental rules, placing also a horizontal line below the cross in addition and subtraction. His use of lines is restricted to indicating that *two* numbers are to be multiplied together, *e.g.* when the three terms of a proposition are $\frac{16}{3}$, $\frac{45}{4}$, $\frac{48}{5}$ he does not write

$$\begin{array}{ccccc} 16 & & 45 & - & 48 \\ \overline{3} & \times & \overline{4} & - & \overline{5} \end{array}$$

to indicate that 3, 15, 48 are to be multiplied together, and also 16, 4, 5; but he represents the procedure by

$$\begin{array}{ccccc} 320 & & 135 \diagdown & & \\ 16 & & 45 & & 48 \\ \overline{3} & \times & \overline{4} & - & \overline{5}\,, \\ & & 20 & & \end{array}$$

where the lines indicate that 3 is to be multiplied by 45, giving 135, which is to be multiplied by 48; and that 5 is to be multiplied by 4, giving 20, which is to be multiplied by 16, giving 320.*

Widman seems to have been the first to use crosses in the chain rule.

§ 200. So far from there having been a slowness of growth in symbolic language between 1450 and 1550, it seems to me that this period is remarkable for the invention of symbols. The cossic signs, the signs + and −, and the sign for square root all came into use in Germany, and probably originated there, during this period. Both Rudolff's *Coss* of 1525 and Stifel's *Arithmetica Integra* of 1544 show a considerable amount of symbolism.

* The cross was used for convenience of printing (as I have used it above) and, as in the present case, it frequently happens that it cannot be so placed that both arms point accurately in the right directions. To obviate this Borgi sometimes raises the second and third fractions, thereby dislocating to some extent the general arrangement. Borgi always begins his multiplications from the extreme denominators.

CONTENTS OF THE PAPER.

PART I.

PART II.

Part III.

Note on the Contents of the Paper.

It may be useful to anyone who refers to this paper to state that Part I. (pp. 1–27) relates to Widman's *Rechenung* (1489), that the greater part of Part III. (viz. pp. 108–140) relates either to Widman's *Rechenung* or to manuscripts that preceded it, while the whole of Part II. (pp. 27–107) relates to works published subsequently to Widman's *Rechenung*.

The signs + and – and the words plus and minus form the principal subject of Parts I. and III. and of considerable portions of Part II. (pp. 33–47, 58–63).

The other subjects to which Part II. relates are: an examination of the questions contained in the various books in which the sign – or an equivalent word occurs in the data (pp. 47–58, 60), the occurrence of the words fusti and tara in German books (pp. 63–77, 89–90), the use of these words by Italian writers (pp. 77–89), and the supposed mention of signs of addition and diminution by Peurbach (pp. 90–107).

The concluding portion of Part III. relates to some recently published extracts from the Bamberg Arithmetic (pp. 132–135), and to the use of the symbols × and = in the rule of three (pp. 135–140).

Widman's use of the cossic notation is described on pp. 22–27: and the Latin Algebra in the Dresden Codex, from which he probably derived + and –, is referred to in various sections from p. 108 to p. 121.

INDEX OF NAMES.

LIST OF BOOKS REFERRED TO.

(The page is where a fuller title is given).

PAPERS IN PERIODICAL PUBLICATIONS REFERRED TO.

MANUSCRIPTS REFERRED TO.

ERRATA.

p. 5, l. 7 from bottom, *for* 240 *read* 210
p. 17, l. 9, *for* 18 *read* 15; l. 12, *for* 8 *read* 81
p. 28, l. 13, *for* 1526 *read* 1527
p. 36, l. 17, *for* bescreib *read* beschreib
p. 37, l. 8 from bottom, *for* § 51 *read* §§ 51–52
p. 42, l. 7, *for* 1543 *read* 1545
p. 43, l. 22, *for* § 86 *read* § 85; l. 29, *for* § 61 *read* §§ 61–62
p. 47, last line, *for* § 69 *read* § 70.
p. 72, l. 17, *for* tara fur *read* thara für
p. 77, l. 6, insert * after Cantor's name; l. 6 from bottom, for Cantor *read Vorlesungen*
p. 77, l. 25, *for* §§ 109–112 *read* §§ 109–123.

ON A NEW CASE OF THE CONGRUENCE $2^{p-1} \equiv 1 \pmod{p^2}$.

By *Dr. N. G. W. H. Beeger.*

In 1913 Mr. W. Meissner discovered the only prime <2000 which satisfies this congruence, *i.e.* the number $p=1093$.*

I now have found that the only prime number between 2000 and 3700, that satisfies the congruence, is $p=3511$.

We have $p=3511=1+2.3^3.5.13$ and $7p=3.2^{13}+1$. Hence

$$3.2^{13} \equiv -1+7p \pmod{p^2},$$

$$3^{135}.2^{1755} \equiv -1+945p \pmod{p^2}\text{............}(1).$$

Now we find by calculation

$$3^{15} \equiv 2021786 \pmod{p^2},$$

and also the residues of 3^{30}, 3^{60}, 3^{120}, 3^{135}. So we find

$$3^{135} \equiv -1+945p \pmod{p^2}\text{..............}(2).$$

From (1) and (2) it follows that

$$2^{1755} \equiv 1 \pmod{p^2}.$$

Here follows a table of the least positive residues of the quotient

$$r \equiv \frac{2^{\xi}-1}{p} \pmod{p},$$

where ξ is the least exponent, $\frac{p-1}{\xi}=\nu$ is called the residue-index.

In the calculation of the table I used the table of residue-indices of Lt.-Col. A. Cunningham.†

p	ν	r	p	ν	r	p	ν	r	p	ν	r
2003	7	1778	2213	1	1490	2393	4	453	2633	2	2
11	5	1055	21	1	2104	9	2	1519	47	2	605
7	6	1292	37	1	1358	2411	5	20	57	16	672
27	1	809	9	2	1839	7	2	1613	9	1	2
9	1	1199	43	1	1163	23	2	2039	63	2	2440
39	2	971	51	3	1721	37	1	760	71	6	1101
53	1	1805	67	1	1405	41	8	383	7	1	765
63	2	489	9	1	1360	7	2	1899	83	1	1697
9	1	1443	73	4	2084	59	1	2160	7	34	2310
81	2	226	81	12	851	67	1	2170	9	12	605
3	1	427	7	6	425	73	4	1696	93	1	866

* *Sitz-Ber. d. Berliner Acad.*, vol. xxxv., p. 663.

† *Messenger of Mathematics*, vol. xxxvii., p. 122.

p	ν	r	p	ν	r	p	ν	r	p	ν	r
7	2	311	93	1	569	7	1	1696	9	1	909
9	72	52	7	2	1291	2503	2	1487	2707	1	2086
99	1	1981	2309	1	1113	21	2	1984	11	2	551
2111	2	1732	11	2	989	31	1	122	3	2	2491
3	48	256	33	1	118	9	1	1394	9	2	2180
29	4	1483	9	1	1403	43	2	1469	29	2	2404
31	1	1773	41	3	1299	9	1	916	31	105	2725
7	2	451	7	3	160	51	2	1889	41	1	768
41	1	1391	51	50	2273	7	1	989	9	3	2496
3	42	789	7	1	1729	79	1	2343	53	2	1570
53	2	1875	71	1	1455	91	2	280	67	6	974
61	2	136	7	2	1003	3	32	2318	77	2	118
79	3	60	81	5	1377	2609	2	2307	89	1	1795
2203	3	1343	3	6	859	17	2	277	91	6	2422
7	2	1389	9	1	136	21	1	2441	7	1	383
2801	2	1454	3041	2	143	3301	5	2385	3527	2	2916
3	1	1262	9	4	1215	7	1	2538	9	4	1444
19	1	1996	61	15	2504	13	4	1217	33	1	3340
33	24	1370	7	1	2617	9	2	702	9	1	3196
7	1	2733	79	2	1993	23	1	1672	41	15	464
43	1	2747	83	1	119	9	2	553	7	1	991
51	1	671	9	4	703	31	15	1504	57	1	1988
7	28	1182	3109	7	1763	43	6	1869	9	2	447
61	1	2190	19	2	1179	7	1	46	71	1	1683
79	2	2075	21	20	1519	59	2	2226	81	1	450
87	2	2256	37	4	2323	61	20	2350	3	2	3123
97	2	2605	63	3	535	71	1	3220	93	2	900
2903	2	8	7	2	87	3	3	2249	3607	6	64
9	1	1762	9	2	1416	89	7	54	13	1	645
17	3	356	81	3	1684	91	30	2158	7	2	213
27	2	1018	7	1	1189	3407	2	2989	23	2	769
39	1	1972	91	58	1771	13	1	1911	31	6	101
53	6	2845	3203	1	32	33	2	927	7	1	3177
7	1	1339	9	2	845	49	8	1495	43	1	517
63	1	491	17	4	2148	57	6	1744	59	1	2168
9	8	1455	21	5	1874	61	1	2115	71	2	1022
71	27	1150	9	3	19	3	6	1973	3	4	2565
99	2	990	51	5	1612	7	1	608	7	1	3004
3001	2	539	3	1	2915	9	1	669	91	1	2983
11	1	2566	7	8	3056	91	1	1740	7	2	2966
9	1	2569	9	3	1773	9	1	1347			
23	2	2521	71	6	2754	3511	2	0			
37	1	2920	99	1	2436	7	1	2163			

CORRESPONDENCES BETWEEN THREE-DIMENSIONAL AND FOUR-DIMENSIONAL POTENTIAL PROBLEMS.

By *Dr. H. Bateman.*

§ 1. It is known that there are certain transformations of co-ordinates which transform a potential problem in a three-dimensional space S_3 into a potential problem in a four-dimensional space S_4. At first sight it may be thought that nothing could be gained by such a transformation, but it happens that in some cases the boundary B_3 in S_4 has a greater degree of symmetry than the corresponding boundary B_2 in S_3, and there is more hope of a solution of the problem being obtained.

Let us consider the transformation*

$$X = x^2 + s^2 - y^2 - z^2, \quad Y = 2(xy - zs), \quad Z = 2(xz + ys) \ldots (1).$$

If $V = F(X, Y, Z) = f(x, y, z, s)$, we have

$$\Box V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial s^2}$$

$$= 4r^2\left(\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} + \frac{\partial^2 V}{\partial Z^2}\right) = 4r^2 \Delta V \ldots\ldots (2),$$

where $\quad r^2 = x^2 + y^2 + z^2 + s^2 = \sqrt{(X^2 + Y^2 + Z^2)} = R.$

A three-dimensional potential function satisfying the potential equation $\Delta V = 0$ is thus transformed into a four-dimensional potential function satisfying the equation $\Box V = 0$. The equation $\Delta\Delta V = 0$ occurring in the theory of elasticity is, however, transformed into $\Box(r^{-2}\Box V) = 0$, so that an elastic problem in S_3 does not correspond to a simple elastic problem in S_4. The equation $\Delta V + k^2 V = 0$, occurring in the theory of vibrations, is transformed into $\Box V + \frac{k^2}{r^2} V = 0$, and again there is a loss of simplicity.

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* *Camb. Phil. Trans.*, vol. xxi. (1910), p. 257; *Proc. Roy. Soc. Edinburgh*, vol. xxxvi. (1917), p. 102.

If V and W are any two functions of X, Y, and Z with continuous partial derivatives, they may be expressed as functions of x, y, z and s by means of equations (1), and we obtain the relation

$$\frac{\partial V}{\partial x}\frac{\partial W}{\partial x}+\frac{\partial V}{\partial y}\frac{\partial W}{\partial y}+\frac{\partial V}{\partial z}\frac{\partial W}{\partial z}+\frac{\partial V}{\partial s}\frac{\partial W}{\partial s}$$
$$=4r^2\left(\frac{\partial V}{\partial X}\frac{\partial W}{\partial X}+\frac{\partial V}{\partial Y}\frac{\partial W}{\partial Y}+\frac{\partial V}{\partial Z}\frac{\partial W}{\partial Z}\right)\text{......}(3).$$

A corresponding relation is obtained by putting $V=W$.

These relations tell us that if $\frac{\partial V}{\partial N}$ denotes the partial derivative of V in a direction normal to a surface $W=$ constant in S_3, and if $\frac{\partial V}{\partial n}$ denotes the partial derivative of V in a direction normal to the corresponding variety $W=$ constant in S_4, then

$$\frac{\partial V}{\partial n}=2r\frac{\partial V}{\partial N}\text{......................}(4).$$

Thus, if we know the value of either V or $\frac{\partial V}{\partial N}$ over the surface $W=$ constant in S_3, we also know the value of either V or $\frac{\partial V}{\partial n}$ over the corresponding variety $W=$ constant in S_4. This means that two important types of potential problems are transformed into similar potential problems in the higher space.

Since $\frac{1}{R}=\frac{1}{r^2}$, a hydrodynamical source or electric pole at the origin of co-ordinates in S_3 corresponds to a hydrodynamical source or electric pole at the origin of co-ordinates in S_4.

To study the geometrical relations it is convenient to write

$$\left.\begin{array}{ll} x=r\cos\theta\cos\phi, & y=r\sin\theta\cos\psi \\ s=r\cos\theta\sin\phi, & z=r\sin\theta\sin\psi \\ r\cos\theta=\xi, & r\sin\theta=\eta \end{array}\right\}\text{.........}(5),$$

then

$$X=r^2\cos 2\theta,\quad Y=r^2\sin 2\theta\cos(\phi+\psi),\quad Z=r^2\sin 2\theta\sin(\phi+\psi),$$
$$X=\xi^2-\eta^2,\qquad \sqrt{(Y^2+Z^2)}=2\xi\eta,\qquad R=\xi^2+\eta^2.$$

A point P with co-ordinates (X, Y, Z) in the space S_3 corresponds to a set of points p lying on a circle (p) in the space S_4. To see this we notice that the quantities r, θ and $\phi+\psi=\alpha$ are constant for the points p, and so these points lie on the hypersphere $r^2=$ constant and also on the plane whose equations are

$$\left.\begin{aligned} x\sin\theta\cos\alpha+s\sin\theta\sin\alpha &= y\cos\theta \\ x\sin\theta\sin\alpha-s\sin\theta\cos\alpha &= z\cos\theta \end{aligned}\right\}\text{.........}(6).$$

The circle (p) is thus of radius r, and it has its centre at the origin.

Let us now calculate the integral of the elementary four-dimensional potential function

$$\frac{1}{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2+(s-s_0)^2}\text{.........}(7)$$

round the circle (p_0) corresponding to the point (X_0, Y_0, Z_0).

Writing

$$\left.\begin{aligned} x_0=r_0\cos\theta_0\cos\phi_0,\quad & y_0=r_0\sin\theta_0\cos\psi_0 \\ s_0=r_0\cos\theta_0\sin\phi_0,\quad & z_0=r_0\sin\theta_0\sin\psi_0 \\ \phi_0+\psi_0 &=\alpha_0 \end{aligned}\right\}\text{......}(8),$$

multiplying the above expression by $r_0 d\phi_0$ and integrating, keeping r_0, θ_0 and α_0 constant, we obtain

$$\frac{2\pi r_0}{[(X-X_0)^2+(Y-Y_0)^2+(Z-Z_0)^2]^{\frac{1}{2}}}\text{..........}(9).$$

A uniform electric charge round the circle (p_0) thus corresponds to an electric pole at the point (X_0, Y_0, Z_0). Combining this result with that embodied in formula (4), we see that the total electric charge on a surface in S_3 is proportional to the total electric charge on the corresponding variety in S_4. There is equality between the two electric charges when we adopt a suitable unit of electric charge in S_4.

A surface of revolution round the axis of X in S_3 may be represented by an equation of type

$$F(X, R)=0 \text{ ....}(10),$$

and is seen to correspond to a variety of revolution of type

$$F(\xi^2-\eta^2,\ \xi^2+\eta^2)=0 \text{ ....}(11).$$

If we write $\Pi = \sqrt{(Y^2+Z^2)}$, it is easy to see that the (X, Π) plane is mapped conformally on the (ξ, η) plane by the parabolic substitution

$$X = \xi^2 - \eta^2, \quad \Pi = 2\xi\eta \quad \ldots\ldots\ldots\ldots\ldots(12).$$

The angles which a curve in the (ξ, η) plane makes with the axes of ξ and η are consequently equal to the angles which the corresponding curve in the (X, Π) plane makes with the axis of X.

A diamond or rhombus in the (ξ, η) plane, with the axes of ξ and η as diagonals, corresponds to a figure in the (X, Π) plane consisting of two parabolic arcs meeting on the axis of X at an obtuse angle at one vertex and at an acute angle at the other, the two angles being supplementary. After a revolution about the axis of X this figure gives a balloon-shaped figure.

A balloon-shaped figure may also be obtained by taking as the figure in the (ξ, η) plane two circular arcs meeting in a small acute angle on the axis of ξ. In this case the nose of the balloon is flat instead of being pointed.

The hydrodynamical problems of irrotational motion associated with a dirigible balloon may then be transformed into the four-dimensional hydrodynamical problems associated with the figure formed by the double revolution of either a rhombus or two circular arcs.

In dealing with hydrodynamical problems associated with a surface of revolution, it is sometimes convenient to introduce a current function Ω connected with the velocity potential Ω by relations of type

$$\frac{\partial V}{\partial X} = \frac{1}{\Pi}\frac{\partial \Omega}{\partial \Pi}, \quad \frac{\partial V}{\partial \Pi} = -\frac{1}{\Pi}\frac{\partial \Omega}{\partial X} \quad \ldots\ldots\ldots(13).$$

When V and Ω are considered as functions of ξ and η, the corresponding relations are

$$2\xi\eta\frac{\partial V}{\partial \xi} = \frac{\partial \Omega}{\partial \eta}, \quad 2\xi\eta\frac{\partial V}{\partial \eta} = -\frac{\partial \Omega}{\partial \xi} \quad \ldots\ldots\ldots(14).$$

This transformation is useful for problems in which there is complete symmetry, the differential equations satisfied by V and Ω are then

$$\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial \Pi^2} + \frac{1}{\Pi}\frac{\partial V}{\partial \Pi} = 0 \quad \ldots\ldots\ldots\ldots\ldots(15),$$

$$\frac{\partial^2\Omega}{\partial X^2}+\frac{\partial^2\Omega}{\partial\Pi^2}-\frac{1}{\Pi}\frac{\partial\Omega}{\partial\Pi}=0 \quad\ldots\ldots\ldots\ldots\ldots(16),$$

$$\frac{\partial^2 V}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial V}{\partial\xi}+\frac{\partial^2 V}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial V}{\delta\eta}=0 \quad\ldots\ldots\ldots(17),$$

$$\frac{\partial^2\Omega}{\partial\xi^2}-\frac{1}{\xi}\frac{\partial\Omega}{\partial\xi}+\frac{\partial^2\Omega}{\partial\eta^2}-\frac{1}{\eta}\frac{\partial\Omega}{\partial\eta}=0 \quad\ldots\ldots\ldots(18).$$

If
$$V=\frac{1}{\sqrt{\{(X-a)^2+\Pi^2\}}},$$
we have
$$\Omega=\frac{X-a}{\sqrt{\{(X-a)^2+\Pi^2\}}},$$
and the corresponding functions in S_4 are

$$V=\frac{1}{[(\xi^2+\eta^2)^2-2a(\xi^2-\eta^2)+a^2]^{\frac{1}{2}}}$$

and
$$\Omega=\frac{\xi^2-\eta^2-a^2}{[(\xi^2+\eta^2)^2-2a(\xi^2-\eta^2)+a^2]^{\frac{1}{2}}}.$$

Other pairs of functions may be derived from these by expanding V and Ω in ascending powers of a and taking the coefficients of a^n.

The case in which a potential function is of form $W(X, \Pi)\cos\alpha$ or $W(\xi^2-\eta^2, 2\xi\eta)\cos(\phi+\psi)$ may be treated by means of the well-known theorem that if $V(X, \Pi)$ is a solution of $\Delta V=0$, then $W=\frac{\partial V}{\partial\Pi}$ is a solution of

$$\frac{\partial^2 W}{\partial X^2}+\frac{\partial^2 W}{\partial\Pi^2}+\frac{1}{\Pi}\frac{\partial W}{\partial\Pi}-\frac{1}{\Pi^2}W=0 \quad\ldots\ldots\ldots(19),$$

and $W(X, \Pi)\cos\alpha$ is consequently a potential function of the required type. The theorem derived from this is that if $V(\xi^2-\eta^2, 2\xi\eta)$ is a solution of $\Box V=0$, then the function V satisfies (17), and the function

$$W=\frac{1}{\xi^2+\eta^2}\left(\eta\frac{\partial V}{\partial\xi}+\xi\frac{\partial V}{\partial\eta}\right) \quad\ldots\ldots\ldots\ldots(20)$$

is a solution of

$$\frac{\partial^2 W}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial W}{\partial\xi}-\frac{1}{\xi^2}W+\frac{\partial^2 W}{\partial\eta^2}+\frac{1}{\eta}\frac{\partial W}{\partial\eta}-\frac{1}{\eta^2}W=0\ldots(21),$$

and consequently $W\cos(\phi+\psi)$ is a four-dimensional potential function. The symmetrical functions and the functions just

considered are those which are needed for the solution of the hydrodynamical problems arising when a solid moves steadily through a fluid with motions of translation and rotation. It may be noticed also that the differential equation (21) is satisfied by an expression of type

$$W = \frac{\partial^2 V}{\partial \xi \partial \eta} \quad \text{.....................(22)},$$

where V is a solution of (17). This provides us with another method of deriving a potential function of type $W \cos(\phi + \psi)$ from a symmetrical potential function V.

Our transformation can also be used to obtain a correspondence between solutions of Laplace's equation by making use of the fact that when V satisfies $\square V = 0$, the function

$$U = \int_{-\infty}^{\infty} V ds \quad \text{.....................(23)}$$

is generally a solution of

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \quad \text{..............(24)}.$$

Let us consider the case when $V = F(X, \Pi)$ depends only on X and Π. Writing $\rho^2 = y^2 + z^2$, we have

$$U = \int_{-\infty}^{\infty} F\{x^2 + s^2 - \rho^2,\ 2\rho\sqrt{(x^2 + s^2)}\}\, ds \text{......(25)}.$$

In particular we have a potential function

$$U = \int_{-\infty}^{\infty} \frac{ds}{(x^2 + s^2 + \rho^2)^{n+1}} P_n\left(\frac{x^2 + s^2 - \rho^2}{x^2 + s^2 + \rho^2}\right) \text{......(26)},$$

which, when $\rho = 0$, reduces to $\dfrac{\Pi}{x^{2n+1}} \cdot \dfrac{1.3\ldots 2n-1}{2.4\ldots 2n}$. Hence we may deduce the relation

$$\frac{\Pi}{r^{2n+1}} \frac{1.3\ldots(2n-1)}{2.4\ldots 2n} P_{2n}(\cos\theta)$$

$$= \int_{-\infty}^{\infty} \frac{ds}{(x^2 + s^2 + \rho^2)^{n+1}} P_n\left(\frac{x^2 + s^2 - \rho^2}{x^2 + s^2 + \rho^2}\right) \text{......(27)},$$

where $\qquad x = r\cos\theta, \ \rho = r\sin\theta.$

Another correspondence between three and four-dimensional potential problems may be based on the transformation

$$X=\frac{x}{s+r},\quad Y=\frac{y}{s+r},\quad Z=\frac{z}{s+r}\quad\text{.........(28)},$$

where $$r^2=x^2+s^2+y^2+z^2.$$

In this case a point P (X, Y, Z) in S_3 corresponds to points p lying on a line $\bar{p}$ in the space S_4 and passing through the origin. A surface in S_3 thus corresponds to a conical variety in S_4.

§ 2. As a first example of the use of our first transformation let us derive the potential function which is constant over the spheroid

$$\frac{R+X}{2a^2}+\frac{R-X}{2b^2}=1\quad\text{................(29)}$$

from the potential function which is constant over the four-dimensional ellipsoidal variety of revolution

$$\left.\begin{aligned}\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}&=1\\ \textit{i.e.}\qquad \frac{x^2+s^2}{a^2}+\frac{y^2+z^2}{b^2}&=1\end{aligned}\right\}\quad\text{..............(30)}.$$

Now in the case of the four-dimensional ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\frac{s^2}{g^2}=1\quad\text{..............(31)},$$

the potential function which is constant over the ellipsoid and zero at infinity is given by the formula

$$V=C\int_\lambda^\infty\frac{d\theta}{[(a^2+\theta)(b^2+\theta)(c^2+\theta)(g^2+\theta)]^{\frac{1}{2}}}\quad\text{...(32)},$$

where C is a constant and λ is the positive root of the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}+\frac{s^2}{g^2+\lambda}=1\quad\text{.......(33)}.$$

Hence in our case the desired potential function is given by the formula

$$V=C\int_\lambda^\infty\frac{d\theta}{(a^2+\theta)(b^2+\theta)}=\frac{C}{a^2-b^2}\log\frac{a^2+\lambda}{b^2+\lambda}\quad\text{...(34)},$$

where λ is defined by the equation

$$\frac{\xi^2}{a^2+\lambda}+\frac{\eta^2}{b^2+\lambda}=1 \text{ ....(35)},$$

or by the equivalent equation

$$\frac{R+X}{2(a^2+\lambda)}+\frac{R-X}{2(b^2+\lambda)}=1 \text{ ....(36)}.$$

As λ varies this equation represents a series of confocal prolate spheroids, the semi-axes A and B of the typical spheroid being

$$A=\lambda+\tfrac{1}{2}(a^2-b^2), \quad B=\sqrt{\{(\lambda+a^2)(\lambda+b^2)\}} \text{....(37)}.$$

We find that the required root of equation (36) is

$$\lambda=\tfrac{1}{2}[(a^2-b^2-X)^2+\Pi^2]^{\frac{1}{2}}-\tfrac{1}{2}(a^2+b^2-R) \text{.....(38)}.$$

Hence

$$a^2+\lambda=\tfrac{1}{2}(R+R'+2\bar{c}), \quad b^2+\lambda=\tfrac{1}{2}(R+R'-2\bar{c}) \text{...(39)},$$

where R and R' denote the distances of a point P from the foci O, and O' of the spheroid and $2\bar{c}$ denotes the distance between the foci. We thus obtain the usual formula

$$V=C'\log\frac{R+R'+2\bar{c}}{R+R'-2\bar{c}} \text{ ....(40)},$$

where C' is a constant which may be identified with $e/2\bar{c}$, where e is the total charge on the spheroid. Since

$$R^2-R'^2=4\bar{c}X,$$

we may in fact write

$$\frac{R+R'+2\bar{c}}{R+R'-2\bar{c}}=\frac{2X+R-R'}{2X-R+R'}=\frac{X+R+\bar{c}}{X+R'-\bar{c}} \text{....(41)},$$

and the above formula becomes identical with the well-known formula* for the potential of a uniform line charge of total magnitude e and extending from O to O'.

As a second example let us consider the four-dimensional potential function which is constant over the "ring" whose equation is

$$[(\xi-b)^2+\eta^2-c^2][(\xi+b)^2+\eta^2-c^2]=0 \text{......(42)}.$$

* Cf. Abraham-Föppl, *Theorie der Electricität*, Bd. I.

To find this function we shall find it convenient to use toroidal co-ordinates σ, χ, defined by the equations*

$$\xi = \frac{a \sinh \sigma}{\cosh \sigma - \cos \chi}, \quad \eta = \frac{a \sin \chi}{\cosh \sigma - \cos \chi} \quad \ldots\ldots(43).$$

If $V = (\cosh \sigma - \cos \chi)\, U$, where U depends only on σ and χ, it is found that U satisfies the equation

$$\frac{\partial^2 U}{\partial \sigma^2} + \coth \sigma \frac{\partial U}{\partial \sigma} + \frac{\partial^2 U}{\partial \chi^2} + \cot \chi \frac{\partial U}{\partial \chi} = 0 \quad \ldots\ldots(44),$$

and that there are particular solutions of types

$$P_n(\cosh \sigma)\, P_n \cos \chi, \quad Q_n(\cosh \sigma)\, P_n(\cos \chi) \quad \ldots.(45).$$

A convenient expression for a constant potential function in terms of special functions is obtained by using Heine's expansion. We have in fact

$$1 = (\cosh \sigma - \cos \chi) \sum_{n=0}^{\infty} (2n+1)\, Q_n(\cosh \sigma)\, P_n(\cos \chi) \ldots(46).$$

A potential function which agrees in value with this over the loci $\sigma = \sigma_0$ and is zero at infinity is given by the formula

$$V = (\cosh \sigma - \cos \chi) \sum_{n=0}^{\infty} (2n+1) \frac{P_n(\cosh \sigma)}{P_n(\cosh \sigma_0)} Q_n(\cosh \sigma_0)\, P_n(\cos \chi) \quad \ldots\ldots(47).$$

To verify that this expression satisfies the requirements we must first note the geometrical meanings of σ and χ.

Let A and A' be the points whose (ξ, η) co-ordinates are $(a, 0)$, $(-a, 0)$ respectively, and let P be a point with co-ordinates (ξ, η), then

$$\frac{PA'}{PA} = e^{\sigma}, \quad A\hat{P}A' = \chi.$$

At infinity we have $\sigma = 0$, $\chi = 0$, $\cosh \sigma = 1$, $\cos \chi = 1$. The expression for V will clearly be zero if the series converges to a finite value when $\sigma = 0$. Now the series is known to converge when $\sigma = \sigma_0$ and $P_n(\cosh \sigma)$ diminishes in value as σ decreases from σ_0 to zero and is positive all the time, hence

* *Electrical and Optical Wave Motion*, p. 104.

each term in the series when $\sigma=0$ is of the same sign as the corresponding term in the series for $\sigma=\sigma_0$ and is numerically less. When, moreover, $\chi=0$, all the terms in each series are positive; hence, since the series converges when $\sigma=\sigma_0$, it also converges when $\sigma=0$. In the foregoing argument we have made use of the theorem that $P_n(\cosh\sigma)$ increases with σ and is greater than 1, its value when $\sigma=0$. This may be seen at once by expanding the integrand in Laplace's integral

$$P_n(\cosh\sigma)=\frac{1}{\pi}\int_0^\pi(\cosh\sigma+\sinh\sigma\cos\phi)^n d\phi\ldots(48)$$

by the binomial theorem. The terms containing an odd power of $\cos\phi$ vanish on integration, while the terms containing an even power are positive and increase with σ. The fact that $Q_n(\cosh\sigma)$ is positive is seen at once from the well-known integral

$$Q_n(z)=\frac{1}{2^{n+1}}\int_{-1}^{+1}(1-t^2)^n(z-t)^{-n-1}dt\ldots\ldots..(49).$$

The solution of our four-dimensional potential problem may now be used to obtain a solution of the three-dimensional potential problem in which V is constant over the surface

$$(R+b^2-c^2)^2=2b^2(R+X)\ldots\ldots\ldots\ldots..(50),$$

where $b^2-c^2=a^2$. This surface is formed by the revolution of a limaçon round its axis of symmetry. We may in fact write

$$\xi=b+c\cos\theta,\quad \eta=c\sin\theta,$$

$$X=b^2+2bc\cos\theta+c^2(\cos^2\theta-\sin^2\theta)=b^2-c^2+2c\cos\theta(b+c\cos\theta),$$

$$\Pi=2bc\sin\theta+2c^2\sin\theta\cos\theta=2c\sin\theta(b+c\cos\theta).$$

Referred to a pole at the point $(b^2-c^2, 0)$ the polar equation of the meridional curve in the (X, Π) plane is

$$\rho=2c(b+c\cos\theta),$$

and this represents a limaçon. We can thus find a potential function which is constant over the surface formed by the revolution of a limaçon round its axis of symmetry.

This result is not new, for the limaçon may be inverted into a spheroid, and the inverse problem may then be treated with the aid of spheroidal harmonics.

AN ENDLESS SUCCESSION OF THEOREMS AS TO TWO COMPLETE INSCRIBED POLYGONS WITH EQUALLY NUMEROUS VERTICES.

By *E. B. Elliott.*

1. When I encountered them I did not recognize the two following simple theorems and (iv) and (v) below as familiar. It does not appear that well-known facts of Projective Geometry readily afford them; and the application of direct analytical methods to their establishment is more tedious than I should have expected.

(i) *If ABC, $A'B'C'$ are two triangles inscribed in the same conic, the points in which AA', BB', CC' respectively meet the lines connecting the points of intersection* (BB', CC') *and* (BC', CB'), *the points of intersection* (CC', AA') *and* (CA', AC'), *and the points of intersection* (AA', BB') *and* (AB', BA'), *are collinear.* Their straight line will be called the chord IJ of the conic.

(ii) *In the same case, the lines connecting the poles of AA', BB', CC' respectively with the points of intersection* (BC, $B'C'$), (CA, $C'A'$), (AB, $A'B'$) *are concurrent.* Their point of concurrency will be called T.

The two theorems are polar to one another with regard to the conic. They have presented themselves as follows. The covariant pair of lines

$$\frac{(x-ay)(x-a'y)}{a-a'}+\frac{(x-by)(x-b'y)}{b-b'}+\frac{(x-cy)(x-c'y)}{c-c'}=0 \qquad \ldots\ldots(1)$$

of the pencil $x-ay=0$, $x-a'y=0$, ... to the six vertices from an origin O on the conic, is a pair of the involution determined by the two pairs

$$(x-ay)(x-a'y)=0, \quad \frac{(x-by)(x-b'y)}{b-b'}+\frac{(x-cy)(x-c'y)}{c-c'}=0,$$

of which the former pair run to A and A'. The latter pair is a pair of the involution determined by the two pairs

$$(x-by)(x-b'y)=0, \quad (x-cy)(x-c'y)=0.$$

It is also a pair of the involution determined by the two pairs

$$(x-by)(x-c'y)=0, \quad (x-cy)(x-b'y)=0.$$

For its equation may be written

$$\frac{1}{y}\left\{\frac{1}{x-by}-\frac{1}{x-b'y}+\frac{1}{x-cy}-\frac{1}{x-c'y}\right\}=0,$$

and so, pairing differently, is the same as

$$\frac{(x-by)(x-c'y)}{b-c'}+\frac{(x-cy)(x-b'y)}{c-b'}=0.$$

Thus the two points I, J on the conic to which the lines (1) run belong to an involution on the conic of which A, A' are one pair and another pair consists of the points in which the conic is cut by the line joining the point of intersection (BB', CC') to the point of intersection (BC', CB'). Accordingly the line IJ passes through the intersection of AA' and the connector of (BB', CC') and (BC', CB').

We have here separated the three pairs aa', bb', cc' into two pairs and a single pair in one of three possible and precisely similar ways. The other two separations give us, just as above, that IJ also passes through the intersection of BB' with the connector of (CC', AA'), (CA', AC'), and through the intersection of CC' with the connector of (AA', BB'), (AB', BA').

Having thus proved (i), we immediately deduce (ii), with T the pole of IJ, by remarking that $(BC, B'C')$ is the pole of the connector of (BB', CC'), (BC', CB'), and similarly for the other points named. (The figure will be found to be compact when $AA'BB'CC'$ is the order round the conic.)

2. To the two theorems a third may be added, if we remember that two triangles inscribed in one conic are circumscribed to another. For this reason the correlative of theorem (i) gives us a theorem as to our present figure of two inscribed triangles, of which the following is a statement:—

(iii) *If* L_1, M_1, N_1 *are the respective intersections of* BC, CA, AB *with* $B'C'$, $C'A'$, $A'B'$, L_2, M_2, N_2 *the respective intersections of* CA, AB, BC *with* $A'B'$, $B'C'$, $C'A'$, *and* L_3, M_3, N_3 *those of* $C'A'$, $A'B'$, $B'C'$ *with* AB, BC, CA, *the lines from* L_1, M_1, N_1 *respectively to the points of intersection* (M_1N_1, L_2L_3), (N_1L_1, M_2M_3), (L_1M_1, N_2N_3) *are concurrent.*

3. Let us next take two complete inscribed quadrangles $ABCD$, $A'B'C'D'$, definitely choosing A and A', B and B', C and C', D and D' as corresponding vertices. Consider the pair of lines, from an origin O on the circumscribing conic,

$$\frac{(x-ay)(x-a'y)}{a-a'}+\frac{(x-by)(x-b'y)}{b-b'}$$
$$+\frac{(x-cy)(x-c'y)}{c-c'}+\frac{(x-dy)(x-d'y)}{d-d'}=0\ldots(2).$$

By H, K mean the points on the conic to which the lines (2) run, and by S mean the pole of HK.

H, K are a pair of the involution on the conic which is determined by the two pairs of points in which the conic is cut by

$$\frac{(x-ay)(x-a'y)}{a-a'}+\frac{(x-by)(x-b'y)}{b-b'}=0,$$

and by $$\frac{(x-cy)(x-c'y)}{c-c'}+\frac{(x-dy)(x-d'y)}{d-d'}=0.$$

Of these two pairs we have seen that the former consists of the points on the conic which lie on the connector of intersections (AA', BB'), $(AB', A'B)$, *i.e.* on the polar of the intersection $(AB, A'B')$, while the latter consists of the points on the conic which lie on the connector of (CC', DD'), $(CD', C'D)$, *i.e.* on the polar of $(CD, C'D')$. Accordingly HK passes through the intersection of the two connectors which have been specified, and the pole S of HK lies on the connector of the points of intersection $(AB, A'B')$, $(CD, C'D')$, which are the poles of those connectors.

Now there are two other ways of separating the sum of four terms on the left in (2) into two sums of two terms, and there is absolute similarity in all three separations. We have then three intersections of two connectors which lie on HK, and three connectors of two intersections which pass through the pole S, and so have arrived at the following two theorems, of which only the former has to be somewhat cumbrously expressed:—

(iv) *If $ABCD$, $A'B'C'D'$ are two quadrangles inscribed in a conic, and if the respective intersections (BB', CC'), (CC', AA'), (AA', BB') are called α_1, β_1, γ_1, the respective intersections $(BC', B'C)$, $(CA', C'A)$, $(AB', A'B)$ called α_2, β_2, γ_2, the respective intersections (AA', DD'), (BB', DD'), (CC', DD') called α_3, β_3, γ_3, and the respective intersections $(AD', A'D)$,*

$(BD', B'D)$, $(CD', C'D)$ called α_4, β_4, γ_4, then the three intersections $(\alpha_1\alpha_2, \alpha_3\alpha_4)$, $(\beta_1\beta_2, \beta_3\beta_4)$, $(\gamma_1\gamma_2, \gamma_3\gamma_4)$ are collinear. The collinearity is on the line HK.

(v) *In the same case, if BC, $B'C'$ meet in L, CA, $C'A'$ in M, AB, $A'B'$ in N, AD, $A'D'$ in P, BD, $B'D'$ in Q, CD, $C'D'$ in R, then LP, MQ, and NR meet in a point S.*

The figure in (v) will be found to be compact when $AA'BCB'DC'D'$ is the order round the conic.

4. Four other points on the line HK determined by (2), and four other lines through the corresponding S, can be specified in the complete figure. Instead of separating the left of (2) in either of three possible ways into two sums of two terms, separate it, in one of four possible and precisely similar ways, into a single term and a sum of three. The lines OH, OK are a pair of the involution, of which two pairs are given by the separate equations

$$\frac{(x-ay)(x-a'y)}{a-a'} + \frac{(x-by)(x-b'y)}{b-b'} + \frac{(x-cy)(x-c'y)}{c-c'} = 0,$$

$$(x-dy)(x-d'y) = 0,$$

and are also a pair of each one of three other involutions, determined by pairs specified in the same way upon dissociating, instead of (d, d'), from the rest of the letters, first (a, a'), then (b, b'), and then (c, c').

The equations of three terms have been interpreted in § 1. The one written down represents the lines from O to the points I, J on the conic which have been associated with the triangles ABC, $A'B'C'$ in theorem (i)—let us say the points I_4, J_4. The pair of lines OH, OK belongs then to the involution determined by the pairs OI_4, OJ_4 and OD, OD', so that HK passes through the intersection of I_4J_4 and DD'. The other separations tell us in like manner that it also passes through the intersections (I_1J_1, AA'), (I_2J_2, BB'), (I_3J_3, CC'), where I_1J_1, I_2J_2, I_3J_3 are the IJ lines of collinearity of theorem (i) in its application to the pairs of triangles $(BCD, B'C'D')$, $(CDA, C'D'A')$, $(DAB, D'A'B')$, respectively. Consequently:—

(vi) *The line HK for two inscribed quadrangles $ABCD$, $A'B'C'D'$, on which lie the three points specified by linear construction in theorem* (iv), *also contains four other points which can be linearly constructed, namely, the points in which the connectors of corresponding vertices AA', BB', CC', DD'*

respectively meet the lines I_1J_1, I_2J_2, I_3J_3, I_4J_4 of the collinearities of theorem (i) *in its applications to the pairs of inscribed triangles whose vertices remain when we remove in turn A and A', B and B', C and C', D and D' from $ABCD$, $A'B'C'D'$.*

The companion theorem, polar to this one for the conic, may be expressed:—

(vii) *The point S, at which, according to theorem* (v), *the lines LP, MQ, NR are concurrent, is also a common point of the connectors of the poles P_1, P_2, P_3, P_4 of AA', BB', CC', DD' respectively with the points T_1, T_2, T_3, T_4 of the concurrency of theorem* (ii) *applied to the pairs of inscribed triangles specified at the end of* (vi).

5. Now consider two complete n-gons $A_1A_2\ldots A_n$, $B_1B_2\ldots B_n$ inscribed in the same conic. All the $\frac{1}{2}n(n-1)$ connectors of two vertices are regarded as belonging to the figure of a complete n-gon. No definite ordering of passage from a first vertex along a definite side to a definite second vertex, and so on round a circuit, is contemplated. But in the two n-gons to which we are attending every vertex A_r is regarded as having a definite correspondent B_r.

For convenience a notation differing from that used in the preceding particular examples is adopted.

Let

$$(x-a_1y)(x-b_1y)=0,\ (x-a_2y)(x-b_2y)=0,\ \ldots,$$
$$(x-a_ny)(x-b_ny)=0$$

be the pairs of lines from an origin O on the conic to corresponding vertices.

The lines

$$\frac{(x-a_1y)(x-b_1y)}{a_1-b_1}+\frac{(x-a_2y)(x-b_2y)}{a_2-b_2}+\ldots$$
$$+\frac{(x-a_ny)(x-b_ny)}{a_n-b_n}=0 \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

are the parallels through O to lines on any one of which the n pairs intercept segments P_1Q_1, &c., such that

$$P_1Q_1^{-1}+P_2Q_2^{-1}+\ldots+P_nQ_n^{-1}=0.$$

Call the connector of the points where they cut the conic the (n, n) line, and its pole the (n, n) point, for $A_1A_2\ldots A_n$, $B_1B_2\ldots B_n$.

We have established certain facts of collinearity on (n, n) lines, and concurrency at (n, n) points, for the cases $n = 3, 4$. The method which has been adopted, that of utilizing the fact that the left-hand side of (3) may in various ways be looked upon as the sum of the left-hand sides of equations of the same type for smaller values of n, leads at once to general conclusions which may be stated as follows:—

(viii) *Taking* any $r < n$, *separate* any r *of the* n *suffixes* $1, 2, \ldots, n$ *from the complementary* $n - r$, *thus getting a set of* r *A's and* r *corresponding B's, and a complementary set of* $n - r$ *A's and* $n - r$ *corresponding B's. The* (r, r) *line of the former set meets the* $(n - r, n - r)$ *line of the latter set in a point on the* (n, n) *line of* $A_1A_2\ldots A_n$, $B_1B_2\ldots B_n$; *and the line joining the* (r, r) *point of the former set to the* $(n - r, n - r)$ *point of the latter set passes through the* (n, n) *point of* $A_1A_2\ldots A_n$, $B_1B_2\ldots B_n$.

The $(1, 1)$ line of A_1, B_1 is their connector, and their $(1, 1)$ point is its pole with regard to the conic.

The $(2, 2)$ line of A_1A_2, B_1B_2 is EF the connector of the intersections (A_1B_1, A_2B_2), (A_1B_2, A_2B_1), and their $(2, 2)$ point is D the intersection (A_1A_2, B_1B_2) (see § 1).

The $(3, 3)$ line for the inscribed triangles $A_1A_2A_3$, $B_1B_2B_3$, and their $(3, 3)$ point, are the line of collinearity, and the point of concurrency, specified in theorems (i) and (ii), when we change the notation.

The $(4, 4)$ line for the inscribed quadrangles $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, and the $(4, 4)$ point for the same, have been exhibited in theorems (iv), (vi) and (v), (vii) respectively, as a line of collinearity of seven points given by linear constructions, and a point of concurrency of seven lines.

The $(5, 5)$ line for $A_1A_2A_3A_4A_5$, $B_1B_2B_3B_4B_5$ on a conic is a line on which lie five intersections of complementary $(4, 4)$ and $(1, 1)$ lines and ten of complementary $(3, 3)$ and $(2, 2)$ lines. Total 15. At the $(5, 5)$ point there are fifteen concurrencies of lines.

The $(6, 6)$ line for $A_1A_2\ldots A_6$, $B_1B_2\ldots B_6$ is a line on which lie six intersections of $(5, 5)$ and $(1, 1)$ lines, fifteen of $(4, 4)$ and $(2, 2)$ lines, and ten of two $(3, 3)$ lines. Total 31. At the $(6, 6)$ point there are 31 concurrencies.

And generally the (n, n) line of two sets of n points on a conic, numbered as corresponding one to one, is specified as one on which lie $2^{n-1} - 1$ intersections of (r, r) and $(n - r, n - r)$ lines, obtained from two corresponding selections of r and the two complementary selections of $n - r$ of the two

sets of n points. These various (r, r) lines, for values of r less than n, having been previously specified by linear constructions, the geometrical specification is extended to (n, n). Similarly an (n, n) point is specified as one through which pass $2^{n-1}-1$ lines which have been provided by earlier specifications.

6. A few miscellaneous remarks follow.

(*a*) For the actual construction of an (n, n) line we need only two of the $2^{n-1}-1$ intersections of (r, r) and $(n-r, n-r)$ lines which lie on it. The following appears to be the most expeditious succession of linear constructions, $\frac{1}{2}n(n-1)$ in number, by which the (n, n) line of $A_1A_2...A_n$, $B_1B_2...B_n$ can be arrived at. We have in the figure at the outset a number n of $(1, 1)$ lines A_1B_1, A_2B_2, ..., A_nB_n. Construct the $(2, 2)$ lines for (A_1A_2, B_1B_2), (A_1A_3, B_1B_3), ..., (A_1A_n, B_1B_n) from pairs of $(1, 1)$ lines, as sides of harmonic triangles of quadrangles. These are $n-1$ in number. Then construct, as connectors of intersections, a number $n-2$ of $(3, 3)$ lines for $(A_1A_2A_3, B_1B_2B_3)$, $(A_1A_2A_4, B_1B_2B_4)$, ..., $(A_1A_2A_n, B_1B_2B_n)$ from the first and second of the constructed $(2, 2)$ lines with their respectively complementary $(1, 1)$ lines A_3B_3, A_2B_2, the first and third with their complementaries A_4B_4, A_2B_2, and so on. Then construct the $(4, 4)$ lines, $n-3$ in number, for the suffixes $(1\ 2\ 3\ 4)$, $(1\ 2\ 3\ 5)$, ..., $(1\ 2\ 3\ n)$ from the first and second of the $(3, 3)$ lines with their complementaries A_4B_4, A_3B_3, the first and third with their complementaries A_5B_5, A_3B_3, and so on. Continue in this way, till at last there only remains the construction of the one (n, n) line for $(A_1A_2...A_n,$ $B_1B_2...B_n)$ from the $(n-1, n-1)$ lines for $(A_1A_2...A_{n-1},$ $B_1B_2...B_{n-1})$ and $(A_1A_2...A_{n-2}A_n, B_1B_2...B_{n-2}B_n)$ with their respective complementaries A_nB_n and $A_{n-1}B_{n-1}$. Total $(n-1)+(n-2)+...+2+1=\frac{1}{2}n(n-1)$ constructions. Each construction is only a taking and joining of two intersections.

(*b*) If two inscribed n-gons are given us without any assignment of one to one correspondence among vertices, the number of (n, n) lines, and that of (n, n) points, provided by the complete figure is $n!$ The covariant of the two n-ics

$$(x-a_1y)(x-a_2y)...(x-a_ny),\quad (x-b_1y)(x-b_2y)...(x-b_ny)$$

which, when equated to zero, represents all the lines from O to points where (n, n) lines cut the conic is of order $2.n!$, being the product of $n!$ quadratic factors like the left of (3), made integral in a_1, b_1, etc., by factors.

(*c*) The theorem (iii) is isolated, and not one of a succession.

(*d*) The (n, n) lines for $(A_1A_2 \dots A_n, B_1B_2 \dots B_n)$ and $(A_1A_2 \dots A_rB_{r+1} \dots B_n, B_1B_2 \dots B_rA_{r+1} \dots A_n)$, *i.e.* the connectors of the points on the conic which are run to from O by one and the other of the pairs of lines

$$\frac{(x-a_1y)(x-a_2y)}{a_1-a_2}+\dots+\frac{(x-a_ry)(x-b_ry)}{a_r-b_r}$$
$$\pm\left\{\frac{(x-a_{r+1}y)(x-b_{r+1}y)}{a_{r+1}-b_{r+1}}+\dots+\frac{(x-a_ny)(x-b_ny)}{a_n-b_n}\right\}=0$$

are harmonic conjugates with regard to the (r, r) and $(n-r, n-r)$ lines through whose intersection both pass. Correspondingly as to (n, n) points. In particular:

(ix) *The seven harmonic conjugates of the* $(4, 4)$ *point* S *for the two inscribed quadrangles* $(ABCD, A'B'C'D')$ *with regard to the named points on the seven lines* LP, MQ, NR, T_1P_1, T_2P_2, T_3P_3, T_4P_4 *which pass through it, according to* (v) *and* (vii), *are themselves common intersections of sets of seven lines in the extended figure, being the* $(4, 4)$ *points for*

$(BCD'A', B'C'DA)$, $(CAD'B', C'A'DB)$, $(ABD'C', A'B'DC)$, $(BCDA', B'C'D'A)$, $(CDAB', C'D'A'B)$, $(DABC', D'A'B'C)$, $(ABCD', A'B'C'D)$ *respectively.*

(*e*) If from (1), made integral by the factor

$$(a-a')(b-b')(c-c'),$$

we subtract the result of replacing in it a', b', c' by b', c', a', we get a covariant quadratic

$$(aa'+bb'+cc'-a'c-b'a-c'b)x^2+\dots=0,$$

which is recognised (cf. *Mess. of Math.*, vol. xlix., p. 179) as denoting the lines from O to the intersections of the conic with a Pascal line, and thus draw the conclusion:—

(x) *The* $(3, 3)$ *line for* $(ABC, A'B'C')$ *cuts the* $(3, 3)$ *line for* $(ABC, B'C'A')$ *on the Pascal line of the hexagon* $A'AB'BC'C$. *It also cuts the* $(3, 3)$ *line for* $(ABC, C'A'B')$ *on the Pascal line of* $AA'BB'CC'$.

7. The method of proof which has been adopted is inapplicable when the conic on which the points A and B lie

consists of two straight lines, and the theorems as to (n, n) *points* are migatory in such a case. But the theorems (i), (iv), (vi), (viii) still hold with statement unaltered, provided the points A lie on one and the point B on the other of the two lines. If S is the intersection of the two lines, the (n, n) line is now the polar line of S with regard to the n lines $A_1B_1, A_2B_2, \ldots, A_nB_n$, as is readily seen by taking the two lines as axes of reference. We have then linear constructions, most likely not novel, for obtaining polar lines with regard to curves of any degree; for the polar line of a point with regard to $A_1B_1, A_2B_2, \ldots, A_nB_n$ is also its polar line with regard to any n-ic through the $2n$ points A and B.

NOTE ON THE INTEGER SOLUTIONS OF THE EQUATION $Ey^2 = Ax^3 + Bx^2 + Cx + D$.

By *L. J. Mordell,* Manchester College of Technology.

In a short note* published recently, I pointed out that no equation of the form

$$y^2 = x^3 + k \qquad\qquad (1),$$

where $k \neq 0$ is given, has more than a finite number of integer solutions. The same result† also holds for the more general equation

$$Ey^2 = Ax^3 + Bx^2 + Cx + D \qquad\qquad (2),$$

where E, A, B, C, D are given integers, provided the right-hand side has no squared factor in x. The proof of these statements depends upon a theorem by Thue‡ that an equation of the form

$$f(x, y) \equiv ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4 = 1 \ldots (3)$$

has at most a finite number of integer solutions if the left-hand side is not a perfect square in x and y, and also upon some results in the arithmetic theory of the binary quartic. As these results (1) and (2) seem so unexpected, and as the other proof§ of (1) recently noted by Landau and Ostrowski depends upon the theory of ideals, it may be of interest to give a proof of (2) reduced to its simplest form.

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* *Proc. Lond. Math. Soc.*, ser. 2, vol. xviii. (1919), p. v. Records of proceedings, etc.

† As noted on page 3 of my booklet *Three Lectures on Fermat's Last Theorem*, printed by the Cambridge University Press.

‡ *Crelle's Journal*, vol. cxxxv. (1909), p. 303 (a, b, c, d, e need not be integers).

§ *Proc. Lond. Math. Soc.*, ser. 2, vol. xix., p. 276.

Multiplying equation (2) by $81E$, putting $9Ey = y_1$, $3x = x_1$, it takes the form

$$y_1^2 = A_1x_1^3 + 3B_1x_1^2 + C_1x_1 + D_1.$$

Putting now

$$A_1y_1 = \eta/2, \quad A_1x_1 + B_1 = \zeta,$$

it is sufficient to prove that the equation

$$\eta^2 = 4\zeta^3 - g_2\zeta - g_3 \dots\dots\dots\dots\dots\dots (4),$$

where g_2 and g_3 are integers, has at most a finite number of integer solutions when its right-hand side has no squared factor in ζ.

Corresponding to every solution of equation (4), we have a binary quartic

$$(1, 0, -\zeta, \eta, e\between X, Y)^4 \dots\dots\dots\dots\dots\dots (5),$$

where

$$e = g_2 - 3\zeta^2,$$

and with invariants

$$g_2 = e + 3\zeta^2, \quad g_3 = \begin{vmatrix} 1, & 0, & -\zeta \\ 0, & -\zeta, & \eta \\ -\zeta, & \eta, & e \end{vmatrix} = -\zeta e - \eta^2 + \zeta^3,$$

as is clear by noting the value of η^2 in equation (4).

All* these quartics with given invariants g_2, g_3 can be divided into a finite number of classes; that is to say, all the quartics (5), of which there would be an infinite number, if equation (4) had an infinite number of integer solutions, can be derived from a finite number of quartics $f(x, y)$ in (3) with $a, b, c, \dots$ integers by means of the linear unit substitution

$$x = pX + rY, \quad y = qX + sY.$$

This gives on equating terms in X^4 and X^3Y on both sides,

$$f \equiv ap^4 + 4bp^3q + \dots eq^4 = 1 \dots\dots\dots\dots\dots (6),$$

$$r\frac{\partial f}{\partial p} + s\frac{\partial f}{\partial q} = 0;$$

also

$$ps - qr = 1,$$

giving r and s uniquely

$$s = \tfrac{1}{4}\frac{\partial f}{\partial p}, \quad r = -\tfrac{1}{4}\frac{\partial f}{\partial q}.$$

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* This is a particular case of a general theorem by Hermite, *Oeuvres*, T. I, p. 84.

Hence an equation (6) would have an infinite number of integer solutions in p and q, and none of the equations (6) have a squared factor since the cubic (4) has none. By Thue's theorem this is impossible, proving the assertion for equation (2). Equation (1) is of course only a particular case of equation (2).

It seems very likely that the same result holds for equations of the form

$$Ey^2 = Ax^n + Bx^{n-1} + \dots$$

if the right-hand side is not the product of a perfect square in x by a quadratic in x, but I cannot prove it.

Note.—Another proof has since been communicated to the *Lond. Math. Soc.*

EVERY POSITIVE RATIONAL NUMBER IS A SUM OF CUBES OF THREE SUCH NUMBERS.

By *H. W. Richmond.*

1. WARING'S famous statement, "That every positive integer can be expressed as the sum of not more than four squares, or nine cubes, or nineteen fourth powers . . . of other positive integers", inevitably suggests similar results for positive rational numbers. Thus some such numbers can, and others cannot, be expressed by the sum of *two* squares; all can be expressed, in an infinite number of ways, as a sum of squares of *three* rational numbers. Sir T. L. Heath* makes it clear that the essential criteria for distinguishing the two classes date from Fermat. As regards cubes, L. E. Dickson† shews that certain numbers cannot be expressed as the sum of two cubes; further knowledge concerning these is much to be desired. Later (pp. 726–728) is quoted a theorem, due to S. Ryley, that every number is a sum of cubes of *three* rational numbers of doubtful sign; also two proofs (by G. Libri and V. A. Lebesgue) that every positive rational number is the sum of cubes of *four* positive rational numbers. The object of the present paper is to shew how Ryley's solution can be

* *Diophantus* (supplement), Cambridge (1910), pp. 267–273.

† *History of the Theory of Numbers*, pp. 572–578.

used to reduce the minimum number of *positive* cubes from *four* to *three*. Dickson refers also to two independent discoveries of Ryley's theorem by T. Strong and others; I have consulted the original versions of all these papers, except that of Lebesgue, which I cannot discover. In § 3 I have explained the reference in Dickson to the *Leeds Correspondent*.

2. All the solutions named above contain a single arbitrary parameter, which (if for the moment we overlook distinctions of sign) may receive any rational value. Ryley's solution is therefore infinitely more general than Libri's or Lebesgue's. A decomposition of a number into four cubes by Ryley's method would contain *two* arbitrary parameters: one of the cubes could in fact be chosen at random and might be regarded as the second parameter. In the formulæ of Libri and Lebesgue all the cubes are positive if the parameter receives any rational value within a certain range, and the same will be proved true in Ryley's formula.

3. *Historical Note.* Ryley's theorem appeared in the *Ladies' Diary*, 1825, p. 35, as a solution of question (No. 1420)—"Required a general theorem by which a natural number may be divided any number of ways into three rational cubes"—proposed by a gentleman who at the same time announced that he was about to publish by subscription an extensive work on *Diophantine Analysis*. Concerning Ryley's solution the editor wrote, "This problem embraces the first improvement in cubes since the time of Euler, and would have been a very difficult one had the method of solution not been developed in the solutions to the particular case in question 211 of the *Leeds Correspondent*. But the following is very different, and a more simple one than the method alluded to". The *Leeds Correspondent* was a periodical—which, I believe, ceased after three numbers—very similar in scope to the *Ladies' Diary*. Question 211 was a highly artificial problem attributed to Diophantus. To find three numbers x, y, z, such that $x^3+y^3+z^3$, $x^6+y^3+z^3$, $y^6+z^3+x^3$, $z^6+x^3+y^3$ may all be squares. One solution came from Mr. S. Ryley, described as a schoolmaster of Leeds, who saw that if only $x^3+y^3+z^3=1/4$, all the four conditions must be satisfied. He resolved 1/4 into a sum of three cubes, and in the *Ladies' Diary* extends his method to any number. He also comments briefly upon the problem of finding positive cubes.

4. *Ryley's resolution of a number N into three cubes.*

$$N = x^3 + y^3 + z^3 \quad \text{......................(1).}$$

Suppose $\quad x = p - q, \quad z = p + q, \quad y = m - 2p,$

so that

$$x + y + z = m \quad \text{......................(2).}$$

$$N = (m - 2p)^3 + 2p^3 + 6pq^2 = m^3 + 6pq^2 - 6p(m - p)^2,$$

$$36p^2q^2 = 36p^2(m - p)^2 - 6pm^3 + 6pN \text{..........(3).}$$

Let $$6pq = 6p(m - p) - e \quad \text{..................(4).}$$

To obtain a simple special solution, suppose further, after substituting in (3), that

$$e^2 = 6pN, \quad 12ep(m - p) = 6pm^3.$$

Then $$2e(m - p) = m^3,$$

and, eliminating p,

$$6emN = 3Nm^3 + e^3 \quad \text{..................(5).}$$

The unknown quantities m and e in equation (5) can both be expressed as rational functions of a parameter in such a way that m, e, p, x, y, z will have rational values when the parameter is rational. Ryley wrote Nv in place of e, and obtained a solution in terms of a parameter $d = m/v$: it is unfortunate that he thought it necessary to solve out the values of x, y, and z explicitly, for they are unwieldy, while $x + y$, $x + z$, $x + y + z$ are simple.

5. *Formulæ used in this paper.* If we write $e = mt$ in (5), we are led to formulæ which are more convenient to handle; compare those given by Strong (*Dickson*, p. 726). Let

$$e = mt, \quad s = 3N/t^3 \text{............}(A).$$

By (5)

$$x + y + z = m = \frac{6Nt}{t^3 + 3N} = t \times \frac{2s}{s + 1} \text{.........}(B).$$

By (4)

$$3(x + y)(x + z) = e = mt = t^2 \times \frac{2s}{s + 1} \text{........}(C),$$

$$x + z = 2p = \frac{e^2}{3N} = t \times \frac{4s}{(s + 1)^2} \text{......}(D).$$

Hence

$$x + y = \frac{e}{3(x + z)} = t \times \frac{s + 1}{6} \text{.........}(E).$$

Finally, the effect of the restriction imposed upon x, y, z by the assumption becomes clear; for, by (D) and (E),

$$3(x+y)^2(x+z) = \tfrac{1}{3}st^3 = N.$$

Thus

$$N = x^3 + y^3 + z^3 = 3(x+y)^2(x+z) \ldots\ldots\ldots (F),$$

or

$$(x+y+z)^3 = 3(x+y)(x+z)(x+2y+z).$$

The values of x, y, z are here presented in terms of a parameter t and a subsidiary parameter $s = 3N/t^3$. They are rational if t is rational, and not otherwise; for

$$t = e/m = 3(x+y(x+z)/(x+y+z) \ldots\ldots\ldots (G).$$

It is now possible to consider under what conditions x and y and z are positive.

6. *Limits of s and t when N, x, y, z are positive.*

N, x, y, and z being positive, equations (G) and (A) prove that t and s are positive; y is positive when $x+y+z > x+z$, *i.e.*

$$y > 0 \text{ if } s > 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Next, z is positive if $x+y+z > x+y$, or $12s > (s+1)^2$;

$$z > 0 \text{ if } 5+\surd(24) > s > 5-\surd(24) \ldots\ldots\ldots (7),$$

i.e. both y and z are positive if s lie between 1 and $5+\surd(24)$. Lastly, x is positive if

$$(x+y)+(x+z) > x+y+z$$

or

$$s^3 - 9s^2 + 15s + 1 > 0.$$

The left-hand member vanishes for three real values of s, viz:

$$s = 3 + 4\cos 20°,\quad s = 3 + 4\cos 140°,\quad s = 3 + 4\cos 260°,$$

and is positive if

$$s > 3 + 4\cos 20°,$$

or if

$$3 + 4\cos 260° > s > 3 + 4\cos 140°.$$

Thus x and y and z are all positive (i) if s lie between 1 and $3 - 4\sin 10°$, or roughly 1 and 2.306; (ii) if s lie between $3 + 4\cos 20°$ and $5+\surd(24)$, or roughly 6.759 and 9.899. For example, if $N=4$, $t=2$, $s=3/2$, we find

$$4 = (53/150)^3 + (12/25)^3 + (47/30)^3.$$

The positive values of x, y, and z given by these formulæ are not simple numbers; one of them must be considerably larger than the sum of the other two, and if s has a value near one of its limits, another of the three numbers x, y, z is small in comparison with the others. But the theorem is established that:

In the problem of expressing any rational number as a sum of cubes of rational numbers (both when the numbers are restricted to positive values and when they are not so restricted), two cubes may be insufficient, but three cubes always provide an infinity of solutions.

NOTE ON THE BINOMIAL THEOREM.

By *Prof. E. J. Nanson.*

LET
$$n_r = n(n-1)\ldots(n-r+1)/r!$$
$$S_r = 1 + n_1 x + \ldots + n_r x^r,$$

then by differentiation

$$\frac{d}{dx}\{S_r(1+x)^{-n}\} = -(n-r)\,n_r x^r (1+x)^{-n-1},$$

and hence by integration

$$S_r(1+x)^{-n} - 1 = -(n-r)\,n_r \int_0^x x^r (1+x)^{-n-1}\,dx,$$

and therefore

$$(1+x)^n - S_r = (n-r)\,n_r (1+x)^n \int_0^x x^r (1+x)^{-n-1}\,dx.$$

Now if x is positive so is $\int_0^x x^r (1+x)^{-n-1}\,dx$, and therefore $(1+x)^n - S_r$ has the sign of $n(n-1)\ldots(n-r)$. Thus, x being any positive number, if $r < n$, we have

$$(1+x)^n - S_r < (n-r)\,n_r (1+x)^n \int_0^r x^r dx < n_{r+1} x^{r+1} (1+x)^n,$$

whilst if $r = n$ we get

$$(1+x)^n = S_n,$$

and if $r > n$ then $(1+x)^n$ lies between S_{r-1}, S_r.

Thus, x being any positive number, if S_r is taken as an approximation to $(1+x)^n$, the error does not numerically

exceed $n_{r+1}x^{r+1}$ when $r > n$, and when $r < n$ it does not exceed $n_{r+1}x^{r+1}(1+x)^n$.

The truth of the binomial theorem for any exponent immediately follows for the case $0 < x < 1$ and also for the case $x = 1$, provided $\operatorname*{Lt}_{r\to\infty} n_r = 0$, and this is so if $n+1$ is positive.

Next let $-1 < x < 0$ and first suppose $n+1$ to be positive. Then obviously

$$\left|\int_0^x x^r(1+x)^{-n-1}\,dx\right| < (1+x)^{-n-1}\left|\int_0^x x^r\,dx\right|$$
$$< (1+x)^{-n-1}\,|\,x^{r+1}/(r+1)\,|,$$

and hence $\qquad |(1+x)^n - S_r| < (1+x)^{-1}|\,n_{r+1}x^{r+1}|$

for any value of r.

The truth of this inequality, when r is so large that S_{r+1} includes the numerically greatest term in the expansion of $(1+x)^n$, has long been known.

Next suppose $n+1$ to be negative. Then obviously

$$\left|\int_0^x x^r(1+x)^{-n-1}\,dx\right| < \left|\int_0^x x^r\,dx\right| < |\,x^{r+1}/(r+1)\,|,$$

and hence $\qquad |(1+x)^n - S_r| < (1+x)^n|\,n_{r+1}x^{r+1}|$.

Thus when $-1 < x < 0$ the error in taking S_r as an approximation to $(1+x)^n$ does not numerically exceed the product of the next term, viz. $n_{r+1}x^{r+1}$, and $(1+x)^{-1}$ or $(1+x)^n$ according as $n+1$ is positive or negative.

The truth of the binomial theorem for any exponent immediately follows for the case $-1 < x < 0$, and also for the case $x = -1$ if n is positive, because $(1+x)^n\int_0^x x^r(1+x)^{-n-1}\,dx$ is finite if n is positive.

The conclusions in regard to the remainder may be summed up as follows, viz. we have

$$|(1+x)^n - S_r| < A\,|\,n_{r+1}x^{r+1}|,$$

when for any positive value of x the coefficient A has the value $(1+x)^n$, 0, or 1 as $r <=> n$, and for the case $-1 < x < 0$ the coefficient A has the value $(1+x)^{-1}$ or $(1+x)^n$ as $n+1$ is positive or negative. When $n+1$ is zero the remainder is $(-1)^{r+1}x^{r+1}(1+x)^{-1}$.

Melbourne,
June 20, 1921.

AN ELEMENTARY NOTE UPON WARING'S PROBLEM FOR CUBES, POSITIVE AND NEGATIVE.

By *H. W. Richmond.*

1. It is the custom to group together under Waring's name results concerning the expression of given numbers as the sum of powers of other numbers. Waring's original assertion with regard to cubes—that every positive integer can be represented as the sum of cubes of not more than nine other positive integers—made in 1770, was proved in 1909 by Wieferich; no one should omit to read the account of the problem given in Professor G. H. Hardy's Inaugural Lecture before the University of Oxford (Clarendon Press, 1920).

Waring's problem may be extended in two obvious ways—rational numbers may be admitted as well as integers, and negative numbers as well as positive. I believe that the following results, quoted from vol. ii. of Prof. L. E. Dickson's exhaustive *History of the Theory of Numbers*, summarise what is known upon these subjects.

(*a*) Every positive rational number is the sum of four positive rational cubes (p. 727). Every rational number is the sum of three rational cubes, positive or negative (p. 726). Certain numbers cannot be expressed as the sum of two rational cubes (pp. 572–578).

(*b*) Zero cannot be expressed as the sum of three (finite) rational cubes (pp. 545–550): it can be expressed as the sum of four or more integral cubes (pp. 550–561, 563–566).

(*c*) Every integer is the sum of five integral cubes, positive or negative (p. 729). Dickson attributes this to Oltramare, who proposed it as a problem. Solutions by Friocourt, Lemoine, Teilhet, and Franel are to be found on the pages to which Dickson gives references.

It is this result (*c*) that I shall first consider, as a stage of the problem of expressing any given integer N as a sum of cubes of (five or fewer) integers, positive or negative. In what follows a "number" is understood to mean an integer, positive or negative.

2. *Every number N which is a multiple of* 6 ($N = 6m$) *is the sum of the cubes of four numbers whose sum is zero.* Further, *N can be expressed as the sum of four such cubes in just as many ways as $2m$ or $N/3$ can be resolved into three factors (without regard to sign) whose sum is even.*

For if $N/3 = abc$, and $a+b+c$ is even $= 2s$,

$$\left.\begin{aligned} N &= \left(\frac{a+b+c}{2}\right)^3 + \left(\frac{a-b-c}{2}\right)^3 + \left(\frac{-a+b-c}{2}\right)^3 + \left(\frac{-a-b+c}{2}\right)^3 \\ &= s^3 + (a-s)^3 + (b-s)^3 + (c-s)^3 \end{aligned}\right\} \quad \ldots(1),$$

and the statements are established without difficulty.

In particular, since $N/3 = 2m = 2m \times 1 \times 1$ (three factors whose sum is even),

$$N = 6m = (m+1)^3 + (m-1)^3 + (-m)^3 + (-m)^3 \ldots.(2),$$

as is algebraically obvious.

3. *Every number N is the sum of cubes of 5 integers.*

Take any number p such that $p \equiv N \pmod 6$. Since $p^3 - p$ or $(p-1)(p)(p+1)$ is the product of three consecutive numbers, it is divisible by 6, *i.e.* $p \equiv p^3 \pmod 6$. Thus

$$N - p^3 \equiv 0 \pmod 6,$$

and, by § 2,

$$\left.\begin{aligned} N &= p^3 + f^3 + g^3 + h^3 + k^3 \\ \text{where} \quad & f+g+h+k = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(3).$$

Here p can be taken at random from the numbers $N \pm 6l$ (l being an integer); and, when p has been chosen, a finite number of sets of values of f, g, h, k having zero sum can be found. In the solutions of Oltramare's problem, quoted by Dickson, a value (usually small) is given to p, and the special form (2) is used for f, g, h, k.

4. *A special solution for any even value of N.*

An interesting set of solutions is obtained by giving p the value N, so that we have for N an expression

$$\left.\begin{aligned} N &= N^3 + f^3 + g^3 + h^3 + k^3 \\ \text{where} \quad & f+g+h+k = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(4).$$

In order to find f, g, h, k, we have to resolve $\frac{1}{3}(N^3 - N)$ into three factors whose sum is even. If N is an even number, $N+1$ and $N-1$ are odd, and dividing one of the numbers by 3 we have three factors, one even and two odd; their sum is therefore even.

Since every even number is of one of the forms $6m$ or $6m \pm 2$, we may write

$$N = 6m, \quad \tfrac{1}{3}(N^3 - N) = (6m+1)(2m)(6m-1),$$

or

$$N = 6m+2, \quad \tfrac{1}{3}(N^3 - N) = (2m+1)(6m+2)(6m+1), \text{ etc.,}$$

and derive the explicit formulæ

$$\left.\begin{aligned} N &= 6m = (6m)^3 + (5m)^3 + (m+1)^3 + (m-1)^3 - (7m)^3 \\ N &= (6m \pm 2) = (6m \pm 2)^3 + (5m \pm 1)^3 + (m \pm 1)^3 + (m)^3 - (7m \pm 2)^3 \end{aligned}\right\} \quad \ldots\ldots(5).$$

5. *Two special solutions of type* (4) *for any odd value of* N.

N being odd, the numbers $N+1$ and $N-1$ are even; one is divisible by four, and the other by two, not by four. Thus $\frac{1}{3}(N^3 - N)$ has factors a, $2b$, $4c$, where a and b are odd. The sum of these is odd; but we derive factors with an even sum by writing either

$$\tfrac{1}{3}(N^3 - N) = 2a \times 2b \times 2c,$$

or

$$= a \times b \times 8c.$$

Every odd number is one of the forms $12m \pm 1$, $12m \pm 3$, $12m \pm 5$, and we are led in each case to two formulæ similar to (5), one of which is now quoted:

$$\left.\begin{aligned} N &= 12m \pm 3 = (12m \pm 3)^3 + (7m \pm 1)^3 + (5m \pm 1)^3 + (m \pm 1)^3 - (13m \pm 3)^3 \\ N &= 12m \pm 1 = (12m \pm 1)^3 + (7m)^3 + (5m \pm 1)^3 + (m)^3 - (13m \pm 1)^3 \\ N &= 12m \pm 5 = (12m \pm 5)^3 + (13m \pm 5)^3 + (11m \pm 5)^3 - (7m \pm 3)^3 - (17m \pm 7)^3 \end{aligned}\right\} \quad \ldots\ldots(6).$$

6. *Limits to the number of cubes required.*

It has now been shown by (3) that every number N can be expressed, in an infinite number of ways, as the sum of five cubes of a special kind, and the question arises whether this number five cannot be reduced, possibly for all values of N, possibly for certain forms of N. [We have indeed already seen in (1) that, when N is of the form $6m$, only four cubes are necessary]. Now all numbers are of one of the forms $3l$, or $3l \pm 1$, and all cubes are therefore of one of the forms $9l$, or $9l \pm 1$. It follows that a number N of the form $9n + 4$ cannot possibly be the sum of cubes of fewer than four numbers, all four numbers being of the form $3l + 1$. A similar result holds if $N = 9n - 4$. *All* numbers can be expressed by five cubes; *some* numbers cannot be expressed by fewer than four. If we seek a rule true universally for all numbers, the problem is clearly defined: "Can all numbers be expressed as the sum of four cubes, positive or negative"? But other less general results may also be worth studying.

7. *Expressions similar to* (5) *and* (6) *for* $Pm + Q$ *as a sum of four cubes.*

A possible elementary method of investigation is suggested

by equations (5) and (6). In each result the sum of the cubes of five linear functions of a variable m reduces to a linear function of m; cannot a similar result hold good for four (or even three) linear functions? Cannot

$$N=(a+mA)^3+(b+mB)^3+(c+mC)^3+(d+mD)^3\equiv Pm+Q \quad (7)$$

for all values of m? If integer values are found for a, b, c, d, A, B, C, D, all the terms of an arithmetical progression will have been expressed as the sum of four cubes. It is undeniable that some numbers, expressible as the sum of four cubes, may not belong to any such arithmetical progression, and will be overlooked by this method. Nevertheless, the method proves simply and in a convincing manner that the majority of numbers (almost exactly 75 per cent.) can be expressed as the sum of four cubes; for it actually supplies the expressions.

8. *Consideration of equation* (7).

It will be supposed that a, b, c, d, A, B, C, D are integers, so that every integer value of m gives an expression for some number as a sum of four cubes. Further, if any factor r were common to A, B, C, D, it could with advantage be absorbed into the variable m by use of a new variable $m'=rm$, inasmuch as all integer values of m' would lead to integer values of N: A, B, C, D will be assumed to have no factor common to all of them.

By writing $-m$ in place of m, and changing the sign of the whole equation, we see that equation (7) also expresses all numbers $Pm-Q$ as the sum of four cubes

$$N'=(-a+mA)^3+(-b+mB)^3+\ldots=Pm-Q \quad (8),$$

and thus proves that every number $\equiv \pm Q \pmod P$ is a sum of four cubes.

Equation (7) implies that

$$\left.\begin{aligned} A^3+B^3+C^3+D^3&=0\\ aA^2+bB^2+cC^2+dD^2&=0\\ 3\,(a^2A+b^2B+c^2C+d^2D)&=P\\ a^3+b^3+c^3+d^3&=Q \end{aligned}\right\} \quad (9).$$

The first of these readily shews that it is useless to attempt to apply this method to a smaller number of cubes than four. The third shews that P is a multiple of 6. For aA^2 and Aa^2 are both even or both odd; hence the fact that the sum of

the four quantities on the left of the second equation vanishes ensures that the four quantities within the bracket of the third have an even sum. [It may be mentioned, although it is not directly helpful to us, that it is possible for P to vanish: it can only do this for rational values of A, B, C, D, a, b, c, d, when the product $ABCD$ is the square of a rational number.]

The values of N, for which (7) can be satisfied, are subject to a serious limitation, viz. that N cannot be of the form $9n \pm 4$. This will be demonstrated in § 9; after which explicit results as to the representation of other numbers as the sum of four cubes will be obtained.

9. *No number N of the form $9n \pm 4$ is given by equation* (7).

If a number N corresponds to a value m_0 of the variable m, $m - m_0$ may be taken as a new variable; then the number N corresponds to the value zero of the new variable. I suppose that this has been done, and that in equation (7) a number N of the form $9n + 4$ corresponds to the value 0 of m; thus

$$N = 9n + 4 = a^3 + b^3 + c^3 + d^3.$$

As pointed out in § 6, each of the numbers a, b, c, d must be of the form $3l + 1$, and therefore by the second of equations (9)

$$A^2 + B^2 + C^2 + D^2 \equiv 0 \pmod 3.$$

This implies (since all squares $\equiv 0$ or $+1$ to mod 3) either that A, B, C, D are all $\equiv 0 \pmod 3$, or that one of them $\equiv 0$ and the other three all $\equiv \pm 1 \pmod 3$. The former supposition is ruled out by § 8; and the latter (by consideration of the remainders to mod 9) is seen to be incompatible with

$$A^3 + B^3 + C^3 + D^3 = 0.$$

Hence, after a similar treatment of $N = 9n - 4$, we see that

No number N of the form $9n \pm 4$ is included among those given by formula (7).

It is not to be inferred that a number $N = 9n \pm 4$ cannot be expressed as a sum of four cubes; many examples, such as

$$400 = 7^3 + 4^3 + 1^3 + (-2)^3,$$

prove the contrary. But, if $a, b, c, d = 7, 4, 1, -2$, no finite values of A, B, C, D satisfy (7).

10. *A simplified form of equations* (7) *and* (9).

In the remaining sections of this paper it will be shown that, of numbers N not of the form $9n \pm 4$, all those of the forms $9n$, $9n \pm 1$, or $9n \pm 3$, and a very large proportion of

those of the form $9n \pm 2$, can be expressed as the sum of four cubes, and explicit formulæ will be stated. This will be done by means of a simplified special form of equations (7) and (9) in which $A + B$ and $C + D$ vanish. Suppose that

$$A = -B = p, \quad C = -D + r.$$

With these values the first of equations (9) is always satisfied, and we have

$$N = (a + mp)^3 + (b - mp)^3 + (c + mr)^3 + (d - mr)^3 \equiv mP + Q \quad \ldots\ldots(10),$$

provided

$$\left.\begin{aligned} (a + b)\,p^2 + (c + d)\,r^2 &= 0 \\ 3\,(a^2 - b^2)\,p + 3\,(c^2 - d^2)\,r &= P \\ a^3 + b^3 + c^3 + d^3 &= Q \end{aligned}\right\} \ldots\ldots\ldots\ldots(11).$$

If p and r possess any common factor, a new variable may (as in § 8) be introduced in place of m; it is therefore allowable to assume that p and r are relatively prime. By the first of equations (10)

$$a + b = fr^2, \quad c + d = -fp^2 \quad \ldots\ldots\ldots\ldots(12),$$

where f is an integer; and, by substitution in the second equation,

$$P = 3fpr\,[(a - b)\,r - (c - d)\,p] \quad \ldots\ldots\ldots(13).$$

11. *Proofs that all numbers N of the forms $9n$, $9n \pm 3$, $9n \pm 1$, may be expressed as the sum of four integer cubes.*

(i) Suppose $p = r = f = 1$; take $a = 1$, $b = 0$, $c = -1$, $d = 0$; equations (12) are satisfied, (13) shews that $P = 6$ and

$$Q = a^3 + b^3 + c^3 + d^3 = 0.$$

Thus $\quad 6m = (1 + m)^3 + (-m)^3 + (-1 + m)^3 + (-m)^3,$

as already stated in (2).

(ii) Again, suppose $p = 1$, $r = 2$, $f = 1$, then

$$a + b = 4, \quad c + d = -1.$$

Taking $a = 2$, $b = 2$, $c = -1$, $d = 0$, we find that $P = 6$, $Q = 15$,

$$6m + 15 = (2 + m)^3 + (2 - m)^3 + (-1 + 2m)^3 + (-2m)^3 \ldots(14).$$

Every number N, divisible by three, whether even or odd, can be expressed as the sum of four cubes.

Lastly, suppose $p = 1$, $f = 1$, $r =$ successively 3 and 6.

(iii) $p = 1$, $r = 3$, $f = 1$; $a + b = 9$, $c + d = -1$.

Take $a=5$, $b=4$, $c=0$, $d=-1$, then $P=18$, $Q=188$,

$$(5+m)^3+(4-m)^3+(3m)^3+(-1-3m)^3=18m+188=18(m+10)+8 \quad \text{......(15)}.$$

Thus all numbers N of the form $18m+8$ are expressed as the sum of cubes of four numbers; changing signs as in (8) all numbers of the form $18m-8$ are so expressed.

(iv) $p=1$, $r=6$, $f=1$; $a+b=36$, $c+d=-1$.

Take $a=b=18$, $c=-1$, $d=0$; then $P=18$, $Q=2\times 18^3-1$,

$$(18+m)^3+(18-m)^3+(-1+6m)^3+(-6m)^3$$
$$=18m+2=18^3-1=18(m+648)-1 \quad \text{...(16)}.$$

Thus all numbers of the forms $18m-1$ and $18m+1$ are expressed as the sum of four cubes.

The cases now established are $N=6m$, $6m\pm 3$, $18m\pm 1$, $18m\pm 8$, *i.e.*

All numbers of the form $9n$, $9n\pm 1$, $9n\pm 3$, *are expressible as the sum of four cubes.*

12. *Numbers N of the form $9n\pm 2$.*

It only remains to consider numbers N of the form $9n\pm 2$. This involves more difficulty than has hitherto been met with, and it will be necessary to examine equations (10)–(13) more minutely in order to discover and analyse the various possibilities.

Let us consider how a number N of the form $9n\pm 2$ can occur in these formulæ. We have seen that P is a multiple of 6; hence, to mod 6,

$$N\equiv Q\equiv a^3+b^3+c^3+d^3\equiv a+b+c+d\equiv f(r^2-p^2).$$

Unless r is divisible by 3 and p not (or *vice versa*) N is a multiple of 3 and cannot $=9n\pm 2$. Hence we confine ourselves to cases where $r=3s$, and p and f are not divisible by 3;

$$a+b=9fs^2, \quad c+d=-fp^2,$$
$$P=9fps\left[3(a-b)s-(c-d)p\right].$$

But now P is divisible by 9, and also a^3+b^3, so that

$$Q\equiv c^3+d^3 \pmod 9.$$

It follows that c and d both $\equiv 1 \pmod 3$, if $N=9m+2$, and both $\equiv -1 \pmod 3$, if $N=9m-2$; and that $c-d$ is divisible by 3. Finally, therefore, the formulæ required for the case $N=9n\pm 2$ become

$$\left.\begin{array}{l} N=(a+mp)^3+(b-mp)^3+(c+3ms)^3+(d-3ms)^3=Pm+Q \\ c=3c'\pm 1, \quad d=3d'\pm 1 \\ a+b=9fs^2, \quad c+d=-fp^2 \\ P=27fpsT, \text{ where } T=(a-b)s-(c'-d')p \\ \text{Also, } p \text{ and } s \text{ are relatively prime; } p \text{ and } f \text{ are divisible by } 3 \end{array}\right\} \qquad \text{......(17).}$$

It is now possible to tabulate completely the cases in which P has a prescribed value of the form 2^a3^b. P is $27fpsT$, and is known when the separate values of f, p, s and T have been assigned, but these values provide only three (linear) equations to determine the four quantities a, b, c, d. The values of a, b, c, d are in fact indeterminate, and involve linearly an arbitrary parameter; but *this parameter can be merged in the variable m*. *One* set of values of a, b, c, d is all that is wanted when f, p, s, and T are known.

The natural procedure is to select a maximum value for P and discuss all cases in which P is either this maximum or a submultiple. The maximum I have chosen is 1296, so that $fpsT$ is 48 or a factor of 48, since it is clearly better to choose a maximum unduly large rather than one too small. Values selected for s and p form the basis of the tabulation; then f is chosen, and for each value of f it becomes plain whether the ambiguities in (17) should be + or −, and whether T is odd or even; finally, all possible values of T are considered. The results of the work, which is quite straightforward, are summed up in the statements that—

Any odd number N of the form $9n \pm 2$ can be expressed as the sum of four cubes by formula (9), except possibly when $N=324m \pm 25$.

Any even number N of the form $9n+2$ can be expressed as the sum of four cubes by formula (9), except possibly when $N=108m \pm 38$ or $432m \pm 164$.

It is thus proved that seventeen out of eighteen odd numbers and nineteen out of twenty-four even numbers of the form $9n \pm 2$ are certainly expressible as the sum of four cubes. The remaining numbers of the form may be so expressible, as may numbers of the form $9n \pm 4$.

In the table now to be given repetitions are avoided; no more than one formula is applicable to any number N.

13. *Synopsis of results.*

$$N=(a+mp)^3+(b-mp)^3+(c+mr)^3+(d-mr)^3=mP+Q.$$

The values of p, r, a, b, c, d, and P are here tabulated for each formula. The value of Q is of little interest and is suppressed; in its place S, the smallest value of N given by the formula ($N < \frac{1}{2}P$), is shewn, so that each formula proves the possibility of expressing as the sum of four cubes all numbers N for which

$$N = \pm S \pmod{P}.$$

	p	r	a	b	c	d	P	S
$N=3n$	1	1	1	0	−1	0	6	0
	1	2	2	2	−1	0	6	3
$N=9n\pm1$	1	3	5	4	0	−1	18	8
	1	6	18	18	−1	0	18	1
$N=9n\pm2$	1	3	6	3	1	2	54	20
	1	3	10	8	−1	−1	108	2
	1	3	11	7	−1	−1	216	56
	1	3	18	18	−5	1	216	92
	1	3	18	18	−8	4	432	16
	2	3	19	17	−8	8	432	52
	4	3	10	8	−16	−17	432	200
	2	3	5	4	12	−2	54	11
	1	6	18	18	−2	1	54	7
	4	3	5	4	−8	−8	108	29
	4	3	6	3	−8	−8	324	133
	4	9	42	39	−5	−11	324	83

Odd numbers of the form $9n \pm 2$, are all included under

$324n \pm (7, 11, 25, 29, 43, 47, 61, 65, 79, 83, 97,$
$101, 115, 119, 133, 137, 151, 155),$

and the five last formulæ in the above table prove that representation by four cubes is possible in the case of remainders (i) ± 11, ± 43, ± 65, ± 97, ± 119, ± 151; (ii) ± 7, ± 47, ± 61, ± 101, ± 115, ± 155; (iii) ± 29, ± 79, ± 137; (iv) ± 133; (v) ± 83. Only one remainder ± 25 is left uncertain, as asserted in § 11. The statement as to even numbers is similarly verified. It is very desirable on the one hand that a definite conclusion should be reached in the few

cases still unsettled, and on the other it seems probable that the capabilities of the method have not been fully explored, and that it will yield these and further results.

14. *General considerations.*

It is the extreme simplicity of the elementary algebraical identities of which this fairly wide sets of results have been derived that has led to the writing of this short paper. The expressions we have obtained for a number N as the sum of cubes of four numbers a, b, c, d are subject to sundry restrictions, which can only be overlooked so long as the method is easy and successful. One such restriction is mentioned in § 7. Another is proved in § 9, and the numbers $N = 9n \pm 4$, to which our method cannot be applied, are unfortunately the only numbers in which four can be definitely asserted to be the least possible number of cubes. For other forms of N* it is always possible that three cubes can be proved to be sufficient. In all the four-cube representations that have been established by means of equations (10)–(13) a restriction has been imposed that $(a+b)/(c+d)$ shall not only be negative, but shall numerically be the square of a rational fraction r/p. Under such stringent conditions it is by no means improbable that the actual minimum number of cubes will never be obtained; rather it is surprising that a four-cube representation can be established for practically 75 per cent of all integers. With equations (7) and (9), which are less restricted, the difficulty of obtaining prescribed values of P in any systematic manner seems at present a fatal objection.

NOTES ON SOME POINTS IN THE INTEGRAL CALCULUS.

By *G. H. Hardy.*

LV.

On the integration of Fourier series.

1. Prof. W. H. Young, in the course of his researches in the theory of Fourier series, has discovered a very beautiful theorem which is of great importance for the evaluation of definite integrals which contain a periodic factor. The theorem may be stated as follows:

Theorem. *Suppose* (i) *that* $f(x)$ *is summable and periodic, with period* 2π; (ii) *that* $g(x)$ *is of bounded variation in the*

* There is no difficulty in showing that *two* cubes are often inadequate.

interval $(0, \infty)$; *and* (iii) *that the integral*

$$(1) \qquad \int_0^\infty |g(x)|\,dx$$

is convergent. Then the value of the integral

$$(2) \qquad \int_0^\infty f(x)\,g(x)\,dx$$

may be calculated by substituting for $f(x)$ *its Fourier series*

$$\tfrac{1}{2}a_0 + \Sigma\,(a_n \cos nx + b_n \sin nx)$$

and integrating formally term by term; so that

$$(3) \quad \int_0^\infty f(x)\,g(x)\,dx = \tfrac{1}{2}a_0 \int_0^\infty g(x)\,dx$$

$$+ \Sigma \left\{ a_n \int_0^\infty g(x)\cos nx\,dx + b_n \int_0^\infty g(x)\sin nx\,dx \right\}.$$

In particular this is so if (ii*a*) $g(x)$ *is positive and decreases steadily as* x *increases, and* (iii*a*) *the integral*

$$(4) \qquad \int_0^\infty g(x)\,dx$$

is convergent.

If $a_0 = 0$, *the condition* (iii) *or* (iii*a*) *may be replaced by the less exacting condition that* (iii*b*) $g(x)$ *tends to zero when* x *tends to infinity.*

Although everything stated here has been proved by Young, his various publications bearing on the matter* do not, so far as I am aware, contain any quite definite and explicit statement of the theorem as a whole, which is the result of the collation of a number of different passages. And the proof, as presented in his writings, is in any case somewhat intricate. In the first place it involves the assumption of 'Parseval's Theorem' in a very general form. Secondly it depends, in part at any rate, on a general theorem concerning

* W. H. Young: (1) 'On the integration of Fourier series', *Proc. London Math. Soc.*, ser. 2, vol. ix, 1910, pp. 449–462; (2) 'On the theory of the application of expansions to definite integrals', *ibid.*, pp. 463–485; (3) 'On integration with respect to a function of bounded variation', *ibid.*, vol. xiii., 1913, pp. 109–150; (4) 'On the Fourier constants of a function', *Proc. Royal Soc.* (A), vol. lxxxv., 1911, pp 14–24.

In order to obtain the theorem as I have stated it, it is necessary to compare Theorem 6 of (2) and Theorem 2 of (1), together with the extension of this latter theorem to an infinite range of integration [(1), pp. 454–455]. The complete result is fundamental in (4): see in particular pp. 17–18 The simplification in the proof of Theorem 6 of (2), due to the generalised theory of integration, is indicated on pp. 147–148 of (3): the argument here supersedes the 'delicate and lengthy' argument of (2), pp. 475–481.

the integration of series, the proof of which presents considerable difficulty. Of this theorem he has offered two proofs. The first is, as he remarks himself, 'delicate and lengthy'; while the second, which is very much simpler, depends upon his general theory of integration with respect to a monotonic function (or function of bounded variation).

It seems worth while, therefore, to include in these notes a proof which is more direct and presupposes a good deal less.

2. I shall require three lemmas.

(1) *If* (i) $s_m(x)$ *is, for every positive integral value of* m, *a measurable function of* x; (ii) $s_m(x)$ *is bounded for* $a \leq x \leq b$, $m = 1, 2, 3, \ldots$; (iii) $s_m(x)$ *tends to a limit* $s(x)$ *when* $m \to \infty$, *for all, or almost all, values of* x; *and* (iv) $f(x)$ *is summable: then* $s(x)f(x)$ *is summable and*

$$\lim_{m \to \infty} \int_a^b s_m(x) f(x)\, dx = \int_a^b s(x) f(x)\, dx.$$

This is a well-known theorem due to Lebesgue and Vitali.* I follow Young in saying that, when conditions (ii) and (iii) are satisfied, $s_n(x)$ *converges boundedly to* $s(x)$.

(2) *If* $s(x)$ *is of bounded variation in the interval* $(0, 2\pi)$, *then the sum of the first* n *terms of its Fourier series converges boundedly to* $s(x)$.†

This is an immediate consequence of the ordinary theory of Dirichlet's integral, when developed (in Jordan's manner) by means of the Second Theorem of the Mean. A formal proof is given by Young.‡

(3) *If* $g(x)$ *is summable and of bounded variation in the infinite interval* $(0, \infty)$§, *then the series*

$$g(x) + g(x + 2\pi) + g(x + 4\pi) + \ldots$$

is convergent for every positive value of x, *and its sum* $G(x)$ *is summable and of bounded variation in the interval* $(0, 2\pi)$.

$$\text{Let}\quad g(x + 2n\pi) = u_n(x), \quad \int_{x+2n\pi}^{x+2(n+1)\pi} g(t)\, dt = v_n(x),$$

where $0 \leq x \leq 2\pi$. The series $\Sigma v_n(x)$ is (absolutely and uniformly) convergent. Also

* See de la Vallée-Poussin, *Cours d'Analyse*, vol. i. (third edition, 1914), p. 264 (Theorem II.); or Young's paper (2), p. 468 (Theorem 2).

† The limit function is

$$\tfrac{1}{2}\{s(x-0) + s(x+0)\},$$

which differs from $s(x)$ at most at an enumerable set of points.

‡ *l.c.* (2), p. 453.

§ *i.e.*, if the conditions (ii) and (iii) of the main theorem are satisfied.

$$u_n(x) - v_n(x) = \int_{x+2n\pi}^{x+2(n+1)\pi} \{g(x+2n\pi) - g(t)\}\, dt$$

is plainly not greater in absolute value than $2\pi V_n$, where V_n is the total variation of $g(t)$ in the interval

$$x + 2n\pi, \quad x + 2(n+1)\pi.$$

Hence $$\Sigma \{u_n(x) - v_n(x)\},$$

and therefore $\Sigma u_n(x)$, is (absolutely and uniformly) convergent.* Thus $G(x)$ is defined for $0 \leq x \leq 2\pi$. Also

$$|G(x)| \leq |g(x)| + |g(x+2n\pi)| + \ldots,$$

so that $G(x)$ is summable and

$$\int_0^{2\pi} |G(x)|\, dx \leq \int_0^{\infty} |g(x)|\, dx.$$

Finally, if x_1 and x_2 are any two points of the interval $(0, 2\pi)$, we have

$$|G(x_1) - G(x_2)| \leq \Sigma\, |g(x_1 + 2nx) - g(x_2 + 2n\pi)|.$$

Hence if we form one of the sums by means of which the variation of $G(x)$ in $(0, 2\pi)$ is defined, it is less than or equal to a corresponding sum formed for $g(x)$ and the infinite interval $(0, \infty)$. Thus the variation of $G(x)$ in $(0, 2\pi)$ does not exceed that of $g(x)$ in $(0, \infty)$.

3. We can now prove the main theorem. Suppose first that the conditions (i), (ii), and (iii) are satisfied. Then

$$\tfrac{1}{2}a_0 \int_0^{\infty} g(x)\, dx + \sum_1^m \left(a_n \int_0^{\infty} g(x) \cos nx\, dx + b_n \int_0^{\infty} g(x) \sin nx\, dx\right)$$

$$= \frac{1}{2\pi} \int_0^{\infty} g(x)\, dx \int_0^{2\pi} \frac{\sin (m+\frac{1}{2})(x-t)}{\sin \frac{1}{2}(x-t)} f(t)\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(t)\, dt \int_0^{\infty} \frac{\sin (m+\frac{1}{2})(x-t)}{\sin \frac{1}{2}(x-t)} g(x)\, dx,$$

* It is plain that, by a trifling modification of this argument, we can prove the following proposition: if $g(x)$ *is of bounded variation in* $(0, \infty)$, *then the difference where a is a constant, tends to a finite limit when* $n \to \infty$; *so that the series and integral*

$$\Sigma g(n+a) - \int^{n+a} g(x)\, dx,$$

$$\Sigma g(n+a), \quad \int^{\infty} g(x)\, dx$$

converge or diverge together. This is a generalisation of the classical 'Cauchy-Maclaurin' test for the convergence of series. If $g(x)$ is an integral, its variation is

$$\int_0^{\infty} |g'(x)|\, dx.$$

The theorem then reduces to one proved by Bromwich, 'The relation between the convergence of series and that of integrals', *Proc. Lond. Math. Soc.*, ser. 2, vol. vi., pp. 327–338.

the inversion of the order of integration being plainly legitimate. But

$$G_m(t) = \frac{1}{2\pi}\int_0^\infty \frac{\sin(m+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}\, g(x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\sin(m+\frac{1}{2})(x-t)}{\sin\frac{1}{2}(x-t)}\, G(x)\, dx$$

is the sum of the first $m+1$ terms of the Fourier series of $G(t)$, and so, by Lemmas 2 and 3, converges boundedly to $G(t)$. Hence, by Lemma 1,

$$\int_0^{2\pi} f(t)\, G_m(t)\, dt \to \int_0^{2\pi} f(t)\, G(t)\, dt = \int_0^\infty f(t)\, g(t)\, dt$$

when $m \to \infty$. This proves the first part of our theorem.*

4. We have still to consider the second part of the theorem, in which

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = 0,$$

but the integral (1) is not convergent. This case is easily reduced to dependence on that which we have already discussed.

Suppose that the conditions (i), (ii), and (iii b) of the theorem are satisfied, and let

$$\gamma(x) = g(2m\pi) \qquad \{2m\pi \leq x < (2m+1)\pi\}$$

and
$$\bar{g}(x) = \gamma(x) - g(x).$$

It is plain that $\gamma(x)$, and therefore $\bar{g}(x)$, is of bounded variation in $(0, \infty)$. Also

$$\int_{2m\pi}^{2(m+1)\pi} |\bar{g}(x)|\, dx = \int_{2m\pi}^{2(m+1)\pi} |g(2m\pi) - g(x)|\, dx \leq 2\pi V_m,$$

where V_n has the same meaning as in the proof of Lemma 3 (§ 2). Hence the integral

$$\int_0^\infty |\bar{g}(x)|\, dx$$

is convergent.

* It might be thought that it would be simpler to begin by considering the case in which $g(x)$ is monotonic. If so, $G(x)$ is plainly summable and monotonic in $(0, 2\pi)$; and the proof is materially simplified.

There is however a difficulty. If $g(x)$ is summable and of bounded variation in $(0, \infty)$, it must tend to zero; and $g(x) = h(x) - k(x)$, where $h(x)$ and $k(x)$ are positive functions which tend steadily to zero. But the summability of $g(x)$ does not necessarily involve that of $h(x)$ and $k(x)$. The point may be illustrated by the example of the function

$$g(x) = \frac{1}{x} - \frac{1}{n+1} \quad (n \leq x < n+1;\ n = 1, 2, \ldots).$$

Thus $\bar{g}(x)$ satisfies the conditions imposed upon $g(x)$ in the preceding analysis, and so (3) holds when $\gamma(x) - g(x)$ is written for $g(x)$. But

$$\int_0^\infty f(x)\,\gamma(x)\,dx = \sum_0^\infty \int_{2m\pi}^{2(m+1)\pi} f(x)\,\gamma(x)\,dx$$
$$= \sum_0^\infty g(2m\pi) \int_0^{2\pi} f(x)\,dx = 0,$$

and similarly

$$\int_0^\infty \gamma(x)\cos nx\,dx = 0, \quad \int_0^\infty \gamma(x)\sin nx\,dx = 0.$$

Hence (3) also holds when $\gamma(x)$ is substituted for $g(x)$, since every integral which occurs in it vanishes. And hence, finally, (3) holds as it stands.

5. The theorem of § 1 enables us to evaluate, in the form of infinite series, the integrals considered in Notes IV. and IX. If

$$f(x) = \phi(\sin^2 x)$$

we have $\quad f(x) \sim \tfrac{1}{2}a_0 + \Sigma\,(a_{2m}\cos 2mx + b_{2m}\sin 2mx).$

If $g(x)$ is positive and tends steadily to zero, and the integral (4) is convergent, then

$$(5)\quad \int_0^\infty \phi(\sin^2 x)\,g(x)\,dx = \tfrac{1}{2}a_0 \int_0^\infty g(x)\,dx$$
$$+ \Sigma \left\{ a_{2m}\int_0^\infty g(x)\cos 2mx\,dx + b_{2m}\int_0^\infty g(x)\sin 2mx\,dx \right\}.$$

If (4) is divergent, then

$$(6)\quad \int_0^\infty \{\phi(\sin^2 x) - \tfrac{1}{2}a_0\}\,g(x)\,dx$$
$$= \Sigma \left\{ a_{2m}\int_0^\infty g(x)\cos 2mx\,dx + b_{2m}\int_0^\infty g(x)\sin 2mx\,dx \right\}.$$

One of the most interesting applications of the theorem, which is signalised by Young, is to the case in which

$$g(x) = x^{p-1} \quad (0 < p < 1).$$

In this case $g(x)$ has an infinity at the origin, and the analysis requires modification. We may begin by taking

$$\bar{g}(x) = 0\ (0 < x < c), \quad g(x) = x^{p-1}\ (x \geq c),$$

and applying the theorem to $\bar{g}(x)$. We thus justify term-by-term integration over the range (c, ∞). In order to justify

integration over $(0, c)$, we have to impose an additional condition on $f(x)$ in the neighbourhood of the origin. If we suppose that $f(x)$ is of bounded variation in $(0, c)$, its Fourier series converges boundedly in that interval. We may therefore, by Lemma 1, multiply by the summable function x^{p-1} and integrate term-by-term. Combining our results, we see that term-by-term integration over the whole interval $(0, \infty)$ is permissible. We thus obtain Young's formula

$$(7)\quad \int_0^\infty \{f(x)-\tfrac{1}{2}a_0\}\, x^{p-1}dx = \Gamma(p) \sum_1^\infty \frac{a_n \cos\frac{1}{2}p\pi + b_n \sin\frac{1}{2}p\pi}{n^p}:$$

$f(x)$ being any periodic and summable function which has bounded variation in an interval $(0, c)$.

6. As an example of the use of theorems of this character, I add the following very simple deduction of the functional equation of Riemann's Zeta-function.

Suppose that

$$f(x) = \sum_0^\infty \frac{\sin(2m+1)x}{2m+1},$$

so that

$$f(x) = \tfrac{1}{4}\pi \quad \{2k\pi < x < (2k+1)\pi\},$$
$$f(x) = -\tfrac{1}{4}\pi \quad \{(2k+1)\pi < x < 2(k+1)\}.$$

Then (7) becomes

$$\tfrac{1}{4}\pi \sum_0^\infty (-1)^k \int_{k\pi}^{(k+1)\pi} x^{p-1}dx = \Gamma(p) \sin\tfrac{1}{2}p\pi \sum_0^\infty \frac{1}{(2m+1)^{1+p}}.$$

The series on the right-hand side converges to

$$(1-2^{1+p})\,\zeta(1+p)$$

if $R(p) > 0$. That on the left-hand side is

$$\frac{\pi^{1+p}}{4p} \sum_0^\infty (-1)^k \{(k+1)^p - k^p\}.$$

It is convergent if $R(p) < 1$. Further, if $R(p) < 0$, it is equal to

$$\frac{\pi^{1+p}}{2p}(1^p - 2^p + 3^p - \ldots) = \frac{\pi^{p+1}}{2p}(1-2^{p+1})\,\zeta(-p).$$

Writing $1 + p = s$, we obtain

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\tfrac{1}{2}s\pi\, \Gamma(s)\, \zeta(s),$$

the functional equation.

END OF VOL. LI.